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# Totally Factorable Operators.

Johnny Gills Jr

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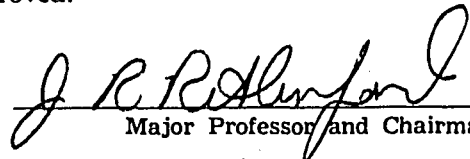
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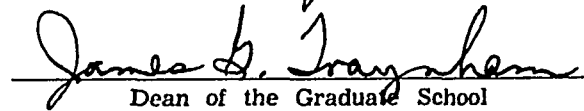
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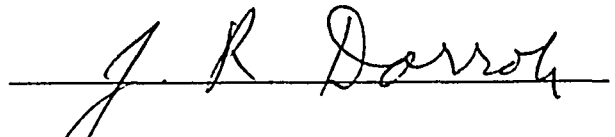
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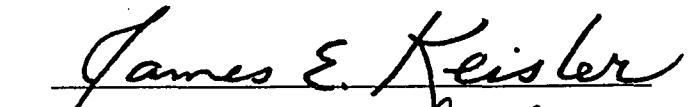
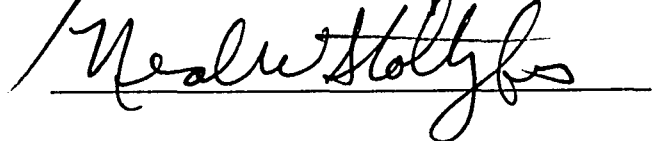
  
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## ABSTRACT

An operator is totally factorable if it factors through every infinite dimensional Banach space. The totally factorable operators offer a new approach to two long outstanding conjectures of A. Grothendieck. The "Big Grothendieck" conjecture: If each bounded linear operator from  $E$  to  $F$  is nuclear either  $E$  or  $F$  is finite dimensional. And, the "Little Grothendieck" conjecture: If every compact operator on  $E$  is nuclear, then  $E$  is finite dimensional. This study to our knowledge is the first of such properties. However, Figiel has studied operators which factor through a subspace of any Banach space, and shown that any compact operator into a Hilbert space meets this requirement.

In Chapter I, the  $S_p$ - and  $D_p$ - spaces are defined. Using a lemma of Figiel we obtain a new result about  $S_p$ -spaces. We later use this result to obtain the machinery crucial to the constructions in Chapter II.

In Chapter II we introduce totally factorable operators. We then almost completely characterize the totally factorable diagonal operators between  $\ell_p$ -spaces. As a by-product of these characterizations we discover that the totally factorable operators between two arbitrary Hilbert spaces are precisely the Hilbert-Schmidt operators. At the end of this chapter we investigate the relationship



between totally factorable operators and the "Little Grothendieck" conjecture. Our main concern is the question: Is every nuclear operator totally factorable? We show that a negative response to this question settles many long outstanding problems in Banach space theory.

In Chapter III, we investigate the composition of totally factorable operators. The major question of the chapter is whether the composition of two totally factorable operators is nuclear. A negative answer to this question implies a positive answer to the "Big Grothendieck" conjecture. We show that in general the composition of two totally factorable operators is Cohen 2-nuclear. We give new proofs of several known results by employing techniques developed in this paper.

In Chapter IV we give three more examples of totally factorable operators (two of which are not compact). We then discuss briefly whether the sum of two totally factorable operators is again a totally factorable operator.

# CHAPTER 0

## FUNDAMENTALS AND NOTATION

Throughout this paper all spaces are infinite dimensional Banach spaces unless otherwise specified. We denote by  $U_E$  the unit ball  $= \{x \in E: ||x|| \leq 1\}$  of  $E$ . We will denote the Banach space of all bounded linear operators from  $E$  into  $F$  by  $L(E, F)$  with norm,

$$||T|| = \sup_{x \in U_E} ||Tx||.$$

By operator, or map, we will always mean a bounded linear operator. A  $T \in L(E, F)$  is called a (weakly) compact operator if  $T(U_E)$  is contained in a (weakly) compact set. The Banach space of all (weakly) compact operators from  $E$  into  $F$  is denoted by  $(W(E, F)) K(E, F)$ .

By  $L_p(K, \Sigma, \mu)$ ,  $1 \leq p \leq \infty$ , we mean the Banach space of equivalence classes of measurable functions  $f$  on some measure space  $(K, \Sigma, \mu)$  for which (if  $p < \infty$ )

$$\int_K |f(x)|^p d\mu(x) < +\infty$$

and with norm,

$$||f|| = \left( \int_K |f(x)|^p d\mu(x) \right)^{1/p}$$

(if  $p = +\infty$ , the space consists of those measurable  $f$  for which  $\|f\| = \text{essential supremum } |f(x)| < +\infty$ ). We shall often omit the measure space  $K$  and the  $\sigma$ -field  $\Sigma$  from the notation and speak simply of an  $L_p(\mu)$ -space.

A subspace is a closed linear subset. Let  $I$  be an index set and  $\{x_i: i \in I\} \subseteq E$ , then by  $[x_i: i \in I]$  we mean the subspace generated by  $\{x_i: i \in I\}$ . If  $M$  is a subspace of  $E$ ,  $J_M^E$  denotes the injection map from  $M$  into  $E$ . By the dual  $E'$ , we mean the Banach space of all continuous linear functionals on  $E$  with norm

$$\|f\| = \sup_{x \in \bigcup_E} |\langle x, f \rangle|.$$

For subspaces  $X \subseteq E$  and  $Y \subseteq E'$ ,

$$X^\perp = \{f \in E': \langle x, f \rangle = 0 \text{ for all } x \in X\}$$

$${}^\perp Y = \{x \in E: \langle x, f \rangle = 0 \text{ for all } f \in Y\}$$

For each  $x \in E$ , let  $J_E x: a \mapsto \langle x, a \rangle$ , then  $J_E$  is an operator from  $E$  into the bidual  $E''$  and  $\|J_E x\| = \|x\|$ . The space  $E$  is reflexive if  $J_E$  is onto. A space  $E$  is reflexive if and only if  $\bigcup_E$  is weakly compact.

For each Banach space  $E$ , we denote by  $I_E$  the identity map on  $E$ . A projection  $P$  is an element of  $L(E, E)$  such that  $P^2 = P$ . A subspace  $M$  of  $E$  is said to be complemented in  $E$  if there is a projection  $P$  on  $E$  with  $P(E) = M$ .

An operator  $T \in L(E, F)$  is called finite, written  $T \in F(E, F)$ , if  $T$  has finite dimensional range. An operator  $T \in F(E, F)$  if and only if there are elements  $a_1, \dots, a_n \in E'$  and elements  $y_1, \dots, y_n \in F$ , such that

$$Tx = \sum_{i=1}^n \langle x, a_i \rangle y_i$$

for  $x \in E$ . We will usually write

$$T = \sum_{i=1}^n a_i \otimes y_i .$$

For each operator  $T \in F(E, F)$

$$\text{Tr}(T) = \sum_{i=1}^n \langle y_i, a_i \rangle$$

where

$$T = \sum_{i=1}^n a_i \otimes y_i .$$

By a biorthogonal system  $(x_i, f_i)$  in  $E$  we mean sequences  $(x_i)$  in  $E$ ,  $(f_i)$  in  $E'$  (called coefficient functionals) such that  $\langle x_i, f_j \rangle = \delta_{ij}$ . A Schauder basis for  $E$  is a biorthogonal system  $(x_i, f_i)$  such that for each  $x \in E$

$$x = \sum_{i=1}^{\infty} \langle x, f_i \rangle x_i ,$$

convergence in the norm of  $E$ . A Schauder basis will be called a basis. A sequence  $(x_i)$  in  $E$  is called a basic sequence if it is a basis for  $[x_i: i \in \mathbb{Z}^+]$ , where  $\mathbb{Z}^+$  denotes the positive integers. Let  $(x_i)$  be a basic sequence with coefficient functionals  $(f_i)$ ; define

$$P_n(x) = \sum_{i \leq n} \langle x, f_i \rangle x_i$$

where  $x \in [x_i: i \in \mathbb{Z}^+]$ . Then  $\sup_n \|P_n\|$  exists and is called the basis constant of  $(x_i)$ . Let  $(x_i)$  and  $(y_i)$  be two basic sequences, then an operator

$$D: [x_i: i \in \mathbb{Z}^+] \rightarrow [y_i: i \in \mathbb{Z}^+]$$

is called a diagonal operator if

$$D\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{\infty} \lambda_i \alpha_i x_i$$

for some sequence  $(\lambda_i)$  of scalars and every

$$x = \sum_{i=1}^{\infty} \alpha_i x_i \in [x_i: i \in \mathbb{Z}^+].$$

We will have need of the following special case of [5, theorem 5, p. 93].

**Theorem 0.1** Let  $E$  be an infinite dimensional normed space. Then there is a biorthogonal system  $(x_i, f_i)$  in  $E$

with  $||x_i|| = ||f_i|| = 1$  for each  $i$ . Furthermore,  $(x_i)$  can be chosen to be basic with basis constant  $\leq 2$ .

Let  $E$  be an arbitrary Banach space.

$$1 \leq p \leq +\infty \quad \text{and} \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Definitions 0.2:

(a)  $\ell_p$  ( $p \neq +\infty$ ) is the Banach space of  $p$ th power summable scalar sequences with norm,

$$||(\lambda_i)||_p = \left( \sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p}.$$

(b)  $\ell_{\infty}$  is the Banach space of all bounded scalar sequences with norm,

$$||(\lambda_i)||_{\infty} = \sup_i |\lambda_i|.$$

(c)  $c_0$  is the subspace of  $\ell_{\infty}$  consisting of those sequences which converge to zero.

(d)  $\ell_p^n$  is the Banach space of all ordered  $n$ -tuples with norm,

$$||(\lambda_i)_{i=1}^n||_p = \begin{cases} \left( \sum_{i=1}^n |\lambda_i|^p \right)^{1/p} & \text{for } p \neq +\infty \\ \max |\lambda_i| & \text{for } p = \infty \end{cases}$$

(e)  $\ell_p(E)$  is the Banach space of sequences  $(x_i)$  in  $E$ , such that  $(\|x_i\|) \in \ell_p$  and norm,

$$\alpha_p(x_i) = \|(\|x_i\|)\|_p.$$

Element of  $\ell_p(E)$  are called absolutely  $p$ -summing sequences.

(f)  $c_0(E)$  is the subspace of  $\ell_\infty(E)$  consisting of those sequences  $(x_i)$  in  $E$  which converge to zero in norm.

(g)  $\ell_p[E]$  is the Banach space of all sequences  $(x_i)$  in  $E$  with  $(\langle x_i, f \rangle) \in \ell_p$  for each  $f \in \bigcup E$ , and norm,

$$\epsilon_p(x_i) = \sup_{f \in \bigcup E} \|(\langle x_i, f \rangle)\|_p.$$

(h)  $\ell_p^*[E']$  is the Banach space of all sequences  $(f_i)$  in  $E'$  with  $(\langle e, f_i \rangle) \in \ell_p$  for all  $e \in E$  and norm,

$$\epsilon_p^*(f_i) = \sup_{e \in \bigcup E} \|(\langle e, f_i \rangle)\|_p.$$

(i)  $\ell_p \langle E \rangle$  is the Banach space of all sequences  $(x_i)$  in  $E$  with  $(\langle x_i, f_i \rangle) \in \ell_1$  for  $(f_i) \in \ell_{p'}[E']$  and norm,

$$\sigma_p(x_i) = \left\{ \left| \sum_{i=1}^{\infty} \langle x_i, f_i \rangle \right| : \epsilon_{p'}(f_i) \leq 1 \right\}.$$

In the case of  $n$ -tuples, the functions  $\epsilon_p$ ,  $\epsilon_p^*$ ,  $\alpha_p$ , and  $\sigma_p$  have the canonical meanings.

Definition 0.3: If  $T \in L(E, F)$ , then  $T$  is called an absolutely  $p$ -summing operator if there is a  $\rho > 0$  such that

$$\alpha_p((Tx_i)_{i=1}^n) \leq \rho \varepsilon_p((x_i)_{i=1}^n)$$

for all finite sets  $\{x_1, \dots, x_n\}$  in  $E$ . The Banach space of all absolutely  $p$ -summing operators from  $E$  into  $F$  is denoted by  $\Pi_p(E, F)$  with norm,

$$\pi_p(T) = \inf \rho,$$

$\rho$  satisfying the above inequality.

Theorem 0.4 If  $p \leq q$ , then  $\Pi_p(E, F) \subseteq \Pi_q(E, F)$  with  $\pi_q(T) \leq \pi_p(T)$ .

We will have need of a factorization result due to Pietsch [34].

Theorem 0.5 If  $T \in \Pi_p(E, F)$  then there exists a probability measure on  $\bigcup_E$ , and operators  $I: E \rightarrow C(\bigcup_E)$ ,  $J: C(\bigcup_E) \rightarrow L_p(\mu)$ , and  $S: JI(E) \rightarrow F$  such that  $T = SJI$ . Furthermore, if  $p = 2$  we may take  $S$  to be defined on  $L_2(\mu)$ .

Definition 0.6: If  $T \in L(E, F)$ , then  $T$  is called a nuclear operator if

$$Tx = \sum_{i=1}^{\infty} \langle x, f_i \rangle y_i,$$



where  $f_i \in E'$ ,  $y_i \in F$  and

$$\sum_{i=1}^{\infty} \|f_i\| \|y_i\| < +\infty.$$

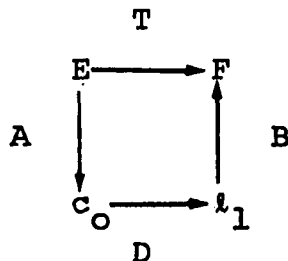
The Banach space of all nuclear operators from  $E$  into  $F$  is denoted by  $\eta(E, F)$  with norm,

$$v(T) = \inf \sum_{i=1}^{\infty} \|f_i\| \|y_i\|$$

where the  $\inf$  is taken over all such representations of  $T$ .

Our next result is a factorization theorem for nuclear operators due to A. Grothendieck [14].

**Theorem 0.7** If  $T \in \eta(E, F)$ , then  $T$  admits the following factorization:



where  $D$  is a diagonal operator corresponding to an element of  $\ell_1$ , and  $A$  and  $B$  are compact. The next result is also due to Grothendieck [14].

**Theorem 0.8** If  $T \in \Pi_2(E, F)$  and  $S \in \Pi_2(F, G)$ , then  $ST \in \eta(E, G)$ .

**Definition 0.9:** Let  $T \in L(E, F)$ , then  $T$  is called a quasi-nuclear if there is an absolutely convergent series

$\sum_{i=1}^{\infty} f_i$  in  $E'$  such that

$$||Tx|| \leq \sum_{i=1}^{\infty} |<x, f_i>| \quad \text{for each } x \in E.$$

We denote by  $Q_n(E, F)$  the quasi-nuclear operators from  $E$  into  $F$ .

The following is due to Pietsch [33].

**Theorem 0.10** Let  $T \in Q_n(E, F)$  and  $\Gamma$  an index set such that  $F \subseteq \ell_{\infty}(\Gamma)$ . Then if  $\psi$  is the injection of  $F$  into  $\ell_{\infty}(\Gamma)$ , then  $\psi T \in Q_n(E, \ell_{\infty}(\Gamma))$ .

**Definition 0.11** Let  $E$  and  $F$  be isomorphic Banach spaces. The distance coefficient of  $E$  and  $F$ ,  $d(E, F)$ , is defined by

$$d(E, F) = \inf ||T|| ||T^{-1}||$$

where  $T: E \rightarrow F$  is an isomorphism.

We are now in a position to define the  $L_p$ -spaces of [26], i.e., those Banach spaces whose finite dimensional subspaces are close to the finite dimensional subspaces of  $L_p(\mu)$ -spaces. It is well known that the  $L_p$ -spaces generalize the  $L_p(\mu)$ - and  $C(K)$ -spaces.

**Definition 0.12** Let  $\lambda \geq 1$  and  $1 \leq p \leq +\infty$ . A Banach space  $E$  is a  $L_p$ -space if for each finite dimensional  $F \subseteq E$  there exists a finite dimensional subspace  $B$  with  $F \subseteq B \subseteq E$

such that  $d(B, \ell_p^n) \leq \lambda$  where  $n = \dim B$ . A Banach space  $E$  is a  $L_p$ -space if  $E$  is a  $L_{p,\lambda}$ -space for some  $\lambda \geq 1$ . We shall use the result that the dual of a  $L_p$ -space is a  $L_{p'}$ -space,  $1 \leq p \leq +\infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , [27].

Two results, essentially due to Grothendieck (proofs may be found in [26]) are:

Theorem 0.13 If  $E$  is a  $L_1$ -space and  $F$  is a  $L_2$ -space, then

$$L(E, F) = \Pi_1(E, F) .$$

Theorem 0.14 If  $E$  is a  $L_\infty$ -space and  $F$  is a  $L_p$ -space for  $1 \leq p \leq 2$ , then  $L(E, F) = \Pi_2(E, F)$ .

In Chapter II, we make repeated use of the following result.

Theorem 0.15 Let  $F$  be an arbitrary Banach space, then any absolutely summing operator  $T: c_0 \rightarrow F$  is nuclear.

Proof: Let  $(e_i)$  denote the usual unit vector basis of  $c_0$ . Then  $\varepsilon_1(e_i) = 1$ , hence  $\sum_{i=1}^{\infty} ||Te_i|| < +\infty$ . If  $x = \sum_{i=1}^{\infty} \langle x, f_i \rangle e_i$  is in  $c_0$ , then  $Tx = \sum_{i=1}^{\infty} \langle x, f_i \rangle Te_i$  where  $f_i$  is the unit vector basis of  $\ell_1$ . Finally,  $\sum_{i=1}^{\infty} ||f_i|| ||Te_i|| = \sum_{i=1}^{\infty} ||Te_i|| < +\infty$  which implies that  $T$  is nuclear.

Remark 0.16 In Chapter III we define the concept of a Banach ideal. Since we will find occasion to use ideal properties, we mention here that  $L$ ,  $K$ ,  $\Pi_p$ , and  $\eta$  are Banach ideals.

**Definition 0.17** A Banach space  $F$  has the  $\lambda$ -extension property if for every  $T_0 \in L(M, F)$ , there exists for each  $E \supseteq M$  an operator  $T \in L(E, F)$  with  $T_0 = T J_M^E$  and  $\|T\| \leq \lambda \|T_0\|$ . An operator  $T_0 \in L(M, F)$  is said to be extendable if for every  $E \supseteq M$ , there exists  $T \in L(E, F)$  such that  $T_0 = T J_M^E$ .

**Theorem 0.18** Every  $L_\infty(\mu)$ -space has the  $\lambda$ -extension property for some  $\lambda$ .

**Theorem 0.19** If  $T_0 \in \Pi_p(M, F)$ , then  $T_0$  is extendable.

**Proof:** See the factorization result 0.5 and 0.18.

## CHAPTER I

### $S_p$ -Spaces

In their paper, "Fully and Completely Nuclear Operators with Applications to  $L_1$ - and  $L_\infty$ -space" [36], Retherford and Stegall introduced the Sufficiently Euclidean spaces. Theorem I.7 of that paper is the following: A Banach space  $E$  is sufficiently Euclidean if and only if each subspace of finite codimension of  $E$  is sufficiently Euclidean. As an immediate consequence of a result first published in [9], we obtain an unexpected improvement of this result. We begin this chapter with the definitions and properties of  $S_p$ - and  $D_p$ -spaces (first introduced in [29]). The  $S_p$ -spaces are natural generalizations of the sufficiently Euclidean spaces.

This chapter is devoted to a very natural generalization, I.21, of Theorem 3.7 of [29]. This result will be a major tool in our construction of totally factorable operators.

Also included in this chapter is a result dual to I.21. This result along with I.21 will enable us to almost completely characterize the diagonal operators between  $\ell_p$ -spaces which are totally factorable.

The last theorem of this chapter is a summary of the major results of this chapter. We then show that the first part of this theorem is in a certain sense, the best possible.

Definition I.1 A Banach space  $E$  is called an  $S_p$ -space,  $1 \leq p \leq \infty$ , if there exists a positive constant  $C$  and sequences of operators  $\{J_n\}$  and  $\{P_n\}$  such that

$$\ell_p^n \xrightarrow{J_n} E \xrightarrow{P_n} \ell_p^n$$

and  $P_n J_n = I_n$ , the identity operator on  $\ell_p^n$ .  $\|J_n\| \leq 1$  and  $\|P_n\| \leq C$  for every  $n$ .

Remark I.2 If  $E$  is an  $S_p$ -space with constant  $C$ , then  $d(J_n(\ell_p^n), \ell_p^n) \leq C$ .

According to [36], a Banach space is sufficiently Euclidean if and only if it is an  $S_2$ -space. We will need the improved version of the "Principle of Local Reflexivity" [20].

Theorem I.3 (The Principle of Local Reflexivity). Let  $X$  be a Banach space (regarded as a subspace of  $X''$ ), let  $E$  and  $G$  be finite dimensional subspaces of  $X''$  and  $X'$ , respectively, and let  $0 < \epsilon < 1$ . Assume that there is a projection  $P$  from  $X''$  onto  $E$  with  $\|P\| = M$ . Then there is a one-to-one operator  $T$  from  $E$  into  $X$  and a projection  $P_0$  from  $X$  onto  $T(E)$  such that

- i.  $Te = e$  for all  $e \in E \cap X$
- ii.  $f(Te) = e(f)$  for all  $e \in E$  and  $f \in G$
- iii.  $\|T\| \|T^{-1}\| \leq 1 + \epsilon$
- iv.  $\|P_0\| \leq M(1 + \epsilon)$

As a consequence to the above result, we obtain the following generalization first given in (29) of Theorem I.4 of [36].

**Theorem I.4** A Banach space is a  $S_p$ -space if and only if  $E'$  is and  $S_{p'}$ -space where  $\frac{1}{p} + \frac{1}{p'} = 1$ .

**Proof:** If  $E$  is an  $S_p$ -space with constant  $C$  and  $\{P_n\}$  and  $\{J_n\}$  the given operators, then clearly the adjoints  $\{P'_n\}$ ,  $\{J'_n\}$  and the same constant  $C$  will suffice for  $E'$ . The converse follows from the Principle of Local Reflexivity. If  $E'$  is an  $S_{p'}$ -space, then  $E''$  is an  $S_p$ -space, so we have the operators  $\{J_n\}$ ,  $\{P_n\}$  and the constant  $C$  given by the definition. Let  $\epsilon=1$ ,  $J_n P_n$  be the projection,  $J_n P_n(E'')$  the finite dimensional subspace of  $E''$ , then we obtain an isomorphism  $S_n: J_n P_n(E'') \rightarrow E$ , and a projection  $Q_n: E \rightarrow E$  such that  $Q_n(E) = S_n J_n P_n(E'')$  with  $\|S_n\| \|S_n^{-1}\| \leq 2$  and  $\|Q_n\| \leq 2C$ ; thus  $\{S_n J_n\}$  and  $\{P_n S_n^{-1} Q_n\}$  are the desired operators and  $4C^2$  the desired constant.

The following lemma and its proof can be found in [9]. First, we need a definition.

**Definition I.5** Let  $(X_i)_{i \in I}$  be a family of Banach spaces, and let  $1 \leq p \leq \infty$ . Let  $\Sigma_p X_i$  denote the space of all functions  $f: I \rightarrow \bigcup_{i \in I} X_i$  such that  $f(i) \in X_i$ , for  $i \in I$ , and the norm of  $f$  defined as follows

$$\|f\|_p = \begin{cases} \left( \sum_{i \in I} \|f(i)\|_{X_i}^p \right)^{1/p} & \text{if } p < \infty \\ \sup \|f(i)\|_{X_i} & \text{if } p = \infty \end{cases}$$

is finite. If  $p = \infty$ , then we require additionally that for every  $\varepsilon > 0$  the set  $\{i \in I : \|f(i)\|_{X_i} > \varepsilon\}$  is finite.

**Lemma I.6** Let  $E$  be a Banach space and  $S \in F(E, E)$ , with  $\text{rank}(S) = m < +\infty$ . Let  $Z$  be a  $r$ -dimensional Banach space. For  $1 \leq p \leq +\infty$ , denote by  $F$  the  $\ell_p$ -sum of  $(mr+1)^2$  copies of  $Z$ . Suppose  $P$  is a projection on  $E$  such that  $\|P\| \leq K$ , and  $d(P(E), F) \leq L$ . Then, there is a projection  $Q$  on  $E$  such that  $QS = SQ = 0$ ,  $\|Q\| \leq KL$ , and  $d(Q(E), Z) \leq L$ .

As an application of the above lemma, we obtain an improvement of Theorem 1.7 in [36]. Theorem 1.7 states the following: A Banach space  $E$  is a  $S_2$ -space if and only if every  $E_0$  (with codimension of  $E_0$  finite) is a  $S_2$ -space. In this case, the projections  $\{P_n\}$  are defined on  $E_0$ . The following result is much stronger. It states that the projections can be chosen to be defined on  $E$  and uniformly bounded independent of the choice of  $E_0$ .

**Theorem I.7** If  $E$  is a  $S_p$ -space with constant  $C$ , then for  $E_0 \subseteq E$  with  $\text{codim}_E E_0 < +\infty$ , there exists sequences  $\{T_n\}$  and  $\{Q_n\}$  such that



$$T_n : \ell_p^n \rightarrow E_0 \subseteq E$$

$$Q_n : E \rightarrow \ell_p^n$$

and  $Q_n T_n = I_n$ , the identity on  $\ell_p^n$ , with  $\|T_n\| \leq 1$  and  $\|Q\| \leq C^3$ .

Proof: Fix  $n$ . Let  $\text{codim } E_0 = m$ , and  $E = E_0 \oplus F$ . Let  $S$  be a projection on  $E$  with  $S(E) = F$ . Let  $K = n(mn+1)^2$ , then  $\ell_p^K$  is equal to the  $\ell_p$ -sum of  $(mn+1)^2$  copies of  $\ell_p^n$ . By I.1 and I.2 there exists  $\{J_K\}$  and  $\{P_K\}$  such that

$$\ell_p^K \xrightarrow{J_K} E \xrightarrow{P_K} \ell_p^K$$

with  $P_K J_K = \text{identity on } \ell_p^K$ ,  $\|J_K\| \leq 1$ ,  $\|P_K\| \leq C$  and  $d(J_K P_K(E), \ell_p^K) \leq C$ . Now applying I.6, there exists a projection  $Q$  on  $E$  such that  $QS = SQ = 0$ ,  $\|Q\| \leq C^2$  and  $d(Q(E), \ell_p^n) \leq C$ . Let  $e \in E$  and  $Q(e) = e_0 + f$ , where  $e_0 \in E_0$  and  $f \in F$ . Now  $Q(e) = e_0$ , since  $f = S(e_0 + f) = S(Q(e)) = 0$ . Therefore,  $Q(E) \subseteq E_0$ . Let  $T$  be an invertible operator from  $\ell_p^n$  to  $Q(E)$  such that  $\|T\| = 1$  and  $\|T^{-1}\| \leq C+1$ . Consider the following diagram:

$$\ell_p^n \xrightarrow{T} Q(E) \xrightarrow{j} E \xrightarrow{Q} Q(E) \xrightarrow{T^{-1}} \ell_p^n$$

where  $j$  is the natural injection. Now, define  $T_n = jT$  and  $Q_n = T^{-1}Q$ . This completes the argument.

**Corollary I.8** If  $E$  is a  $S_p$ -space, then there exists a constant  $C$  such that for every subspace  $E_0$  of finite co-dimension, there exists sequences  $\{F_n\}$  and  $\{Q_n\}$  satisfying:  
 $F_n \subseteq E_0$ ;  $d(F_n, \ell_p^n) \leq C$ ;  $Q_n$  are projections on  $E$ ;  $Q_n(E) = F_n$  and  $\|Q_n\| \leq C$ .

**Proof:** Combine I.2 and I.7.

Every  $L_p$ -space,  $1 \leq p \leq +\infty$  is a  $S_p$ -space. In [36, Corollary 1.6] it is shown that every  $L_p$ -space,  $1 < p < +\infty$  is a  $S_2$ -space, and that  $L_1$ - and  $L_\infty$ -spaces are not  $S_2$ -spaces.

An outstanding contribution to Banach space theory is the following result due to Dvoretzky [8].

**Theorem I.9** For each  $\varepsilon > 0$  and each positive integer  $n$ , there exists an integer  $N(n, \varepsilon)$  such that if  $E$  is any Banach space and  $\dim E > N(n, \varepsilon)$ , then there exists a subspace  $F$  of  $E$  such that  $d(F, \ell_2^n) \leq 1 + \varepsilon$ .

Extending the above conclusion in a natural fashion, we make the following definition:

**Definition I.10** A Banach space  $E$  is a  $D_p$ -space if there exists a constant  $C$  and a sequence  $\{F_n\}$  of subspaces of  $E$  satisfying  $d(F_n, \ell_p^n) \leq C$  for every  $n$ .

**Remark I.11** Every  $S_\infty$ -space is a  $D_p$ -space for all  $p \geq 1$ .

Rephrasing Dvoretzky's result: Every infinite

dimensional Banach space is a  $D_2$ -space where the constant may be chosen to be  $1+\epsilon$  for any  $\epsilon>0$ .

A parallel result to Corollary I.8 for  $D_p$ -spaces is the following.

Corollary I.12 If  $E$  is a  $D_p$ -space, then there exists a constant  $C$  such that for every subspace  $E_0$  with  $\text{codim}_{E_0} E_0 < \infty$ , we can find subspaces  $F_n \subseteq E_0$  and  $d(F_n, \ell_p^n) \leq C$ .

Relationships between  $S_p^-$ ,  $D_p^-$ , and  $L_p$ -spaces can be found in [29].

We now turn our attention to a large collection of spaces which were first introduced in [6].

Definition I.13 A Banach  $E$  has LUST if there is a family  $E_i$   $i \in I$  of finite dimensional spaces, each with unconditional basis constant 1, and a constant  $C \geq 1$  such that for each finite dimensional subspace  $F \subseteq E$ , there is an  $i \in I$  and an operator  $T: E_i \rightarrow E$  such that  $T(E_i) \supseteq F$  and  $\|e\| \leq \|Te\| \leq C\|e\|$  for  $e \in E_i$ , and moreover, for each  $i$  there is an operator  $S: E_i \rightarrow E$  with  $\|e\| \leq \|Se\| \leq C\|e\|$  for  $e \in E$ .

Discussion of the various properties of Banach spaces with LUST can be found in the following papers [6], [10], [18], [21], [25], and [35]. An abbreviated discussion of some of the ideas involving LUST can be found in Section 11 of [35].

For our particular discussion here we only need the following result due to Johnson and Tzafriri [21].

**Theorem I.14** If  $X$  is a subspace of a Banach lattice which is not a  $D_\infty$ -space, then  $X$  contains uniformly complemented  $E_n$  with  $\sup d(E_n, \ell_1^n) < +\infty$  or  $\sup d(E_n, \ell_2^n) < \infty$ . Thus, if  $X$  has LUST, then  $X$  is either a  $S_1^-$ ,  $S_2^-$ , or  $S_\infty$ -space.

Although it is known that not every Banach space has LUST (see [25]), it is still conjectured that every Banach space is an  $S_1^-$ ,  $S_2^-$ , or  $S_\infty$ -spaces! All known examples of Banach spaces not possessing LUST are either an  $S_1^-$ ,  $S_2^-$ , or  $S_\infty$ -space.

We now turn our attention to the problem of presenting two "series properties" which are enjoyed by  $D_p^-$  and  $S_p^-$ -spaces. Finally, we will show how these properties may be an approach to solving the "Little Grothendieck" conjecture: If  $K(E, E) = \eta(E, E)$ , then  $E$  is finite dimensional.

These "series properties" are extensions of a result of [29]. The techniques in the constructions have been used before [17].

**Definition I.15** Let  $P$  and  $Q$  be subspaces of a Banach space  $E$ . The inclination of  $P$  and  $Q$  is defined to be

$$I(P; Q) = \inf\{\|x+y\| : x \in P, \|x\| = 1, y \in Q\}$$

Definition I.16 Let  $(x_i)$  be a sequence in a Banach space  $E$ , then define

$$\theta(x_i)_{i=1}^m = \min_{1 \leq S < m} \{I([x_i; i \leq S]; [x_i, S < i \leq m])\}$$

The next two results are due to Guararii [17].

Theorem I.17 Let  $(x_i)_{i=i_{K-1}+1}^{i_K}$  be a basis for  $P_K$  where  $\{i_K\}_{K=1}^\infty$  is a subsequence of the positive integers,  $i_0 = 0$ . If  $\theta(x_i)_{i=i_{K-1}+1}^{i_K} \geq \alpha > 0$  for each  $K$ , and for any integer  $m$ ,

$$I(P_1 \oplus \dots \oplus P_m; P_{m+1}) \geq \beta_i > 0$$

where  $\inf \beta_i = \beta > 0$ . Then  $(x_i)$  is a basic sequence.

Theorem I.18 Given  $\epsilon > 0$  and a finite dimensional subspace  $P$  of an infinite dimensional Banach space, there exists an infinite dimensional subspace  $Q$  of  $E$  such that  $I(P; Q) > 1 - \epsilon$ .

Theorem I.19

(a) Let  $T: \ell_p^n \rightarrow E_n \subset E$  be an isomorphism with  $\|T\| \|T^{-1}\| \leq C$ , then  $\epsilon_Q(y_i)_{i=1}^n \leq C$  where  $Te_i = x_i$  and  $y_i = x_i / \|x_i\|$ .

(b) Define  $y_i' = \|x_i\| (T^{-1})' e_i'$ , where  $(e_i')$  is the usual unit vector basis of  $\ell_q^n$ , then  $y_i'(y_j) = \delta_{ij}$  and  $\epsilon_p^*(y_i') \leq C$ .

$$(c) \quad \theta(y_i)_{i=1}^n \geq \frac{1}{C}$$

Proof of part (a). We will assume  $||T|| = 1$ . Now,  
 $1 = ||e_i|| = ||T^{-1} x_i|| \leq ||T^{-1}|| ||x_i|| \leq C ||x_i||$ .  
Hence  $1/||x_i|| \leq C$ .

$$\begin{aligned} \varepsilon_q(y_i)_{i=1}^n &= \sup_{f \in U_E} \left( \sum_{i=1}^n |y_i, f|^q \right)^{1/q} \\ &\leq C \sup_{i=1}^n \left( \sum_{i=1}^n |f T(e_i)|^q \right)^{1/q} \\ &\leq C, \quad \text{since } fT \in (\ell_p^n)' \end{aligned}$$

and  $||fT|| \leq 1$ .

Proof of part (b).

$$\begin{aligned} y_i'(y_j) &= ||x_i|| (T^{-1})' e_i'(y_j) \\ &= ||x_i|| e_i'(T^{-1} y_j) \\ &= ||x_i|| / ||x_j|| e_i'(e_j) = \delta_{ij} \end{aligned}$$

If  $x \in [y_i : 1 \leq i \leq n]$  and  $||x|| \leq 1$ , then  $T^{-1}x = (\xi_i)_1^n \in \ell_p^n$  with  
 $||(\xi_i)_1^n||_{\ell_p^n} \leq C$ .

$$\begin{aligned}
\varepsilon_p^*(y_i^!) &= \sup_{\substack{||x|| < 1 \\ x \in [y_i^!]}} \left( \sum_{i=1}^n |y_i^!(x)|^p \right)^{1/p} \\
&= \sup \left( \sum ||k_i|| (T^{-1})^* e_i^!(x) \right)^p)^{1/p} \\
&= \sup \left( \sum ||k_i|| |e_i^! T^{-1} x|^p \right)^{1/p} \\
&\leq \sup \left( \sum |e_i^! T^{-1} x|^p \right)^{1/p} \\
&= \left( \sum_{i=1}^n |\xi_i|^p \right)^{1/p} \leq c .
\end{aligned}$$

Proof of part (c). Let  $1 \leq s < n$ ,  $w \in [x_i : 1 \leq i \leq s]$  and  $z \in [x_i : s < i \leq n]$ . Then  $w = \sum_{i=1}^s a_i T e_i$  and  $z = \sum_{i=s+1}^n a_i T e_i$ .

$$\begin{aligned}
c ||z+w|| &\geq ||T^{-1}|| \left| \left| T \left( \sum_1^n a_i e_i \right) \right| \right| \\
&\geq \left| \left| \sum_1^n a_i e_i \right| \right|_{\ell_p^n} \\
&\geq \left| \left| \sum_1^s a_i e_i \right| \right|_{\ell_p^n} \\
&= ||T|| \left| \left| \sum_1^s a_i e_i \right| \right|_{\ell_p^n} \\
&\geq \left| \left| T \left( \sum_1^s a_i e_i \right) \right| \right| \\
&= ||w|| .
\end{aligned}$$

Hence,  $\theta(y_i) \geq \frac{1}{C}$ .

Corollary I.20. Let  $E$  be a  $D_p$ -space. Then there exists a constant  $C$  such that every finite codimensional subspace  $Q$  contains an  $n$ -tuple  $(y_i)_{i=1}^n$  of unit vectors in  $Q$  with  $\epsilon_q(y_i)_1^n \leq C$ , and  $\theta(y_i)_1^n \geq 1/C$ . Furthermore, the corresponding coefficient functionals  $(y'_i)$  on  $[y_i: 1 \leq i \leq n]$  satisfy

$$\epsilon_p^*(y'_i) \leq C$$

Proof: See I.12 and I.19.

Theorem I.21 For  $1 \leq p \leq \infty$ , let  $E$  be a  $D_p$ -space and  $(a_i) \in c_0$  with  $0 < a_i \leq 1$ , then there exists a basic sequence  $(x_i)$  in  $E$  such that  $\|x_i\| = a_i$  and  $(x_i) \in \ell_q[E]$ . In the case  $q = +\infty$ , we also have  $(x_i) \in c_0[E]$ . Furthermore, there exists  $(f_i) \subseteq [x_i: i \in \mathbb{Z}^+]$  such that  $(a_i f_i) \in \ell_p^*[[x_i]]$  and  $f_i(x_j) = a_i \delta_{ij}$ . Finally in the case  $p = \infty$ ,  $(a_i f_i) \in c_0[[x_i]]$ .

Proof: Select a subsequence  $(i_k)$  of the positive integers satisfying the property: if  $i > i_k$ , then  $a_i \leq 1/2^k$ . Let  $\sigma_k = \{i_{k-1}+1, \dots, i_k\}$ ,  $i_0 = 0$ . By I.20, there exists  $(y_i)_{i \in \sigma_1}$  such that  $\|y_i\| = 1$ ,  $\theta(y_i)_{i \in \sigma_1} \geq \frac{1}{C}$  and  $\epsilon_q(y_i) \leq C$ . Let  $x_i = a_i y_i$ , then  $\theta(x_i) \geq \frac{1}{C}$  and  $\|x_i\| = a_i$ . For purposes of notation, let  $A_1 = (z_i)_1^\infty$  where



$$z_i = \begin{cases} x_i = a_i y_i & i \in \sigma_1 \\ 0, & \text{otherwise} \end{cases}$$

Now,  $\epsilon_q(A_1) = \epsilon_q(x_i)_{i \in \sigma_1} \leq (\sup_{i \in \sigma_1} a_i) \epsilon_q(y_i) \leq C$ . Select a sequence  $(\epsilon_i)$  with  $0 < \epsilon_1 < 1$  and  $\prod (1 - \epsilon_i) = \beta > 0$ . Let  $F_1 = [x_i : i \in \sigma_1]$ . Then by Theorem I.18, there exists a subspace  $Q_1$  of  $E$  such that  $\text{codim}_E Q_1 < \infty$  and  $I(F_1; Q_1) > 1 - \epsilon_1$ . Applying I.20 again there exists  $(y_i)_{i \in \sigma_2}$  in  $Q_1$  with  $\|y_i\| = 1$ ,  $\theta(y_i) \geq \frac{1}{C}$  and  $\epsilon_q(y_i) \leq C$ . If  $A_2 = (z_i)_1^\infty$ , where

$$z_i = \begin{cases} x_i = a_i y_i, & i \in \sigma_2 \\ 0, & \text{otherwise,} \end{cases}$$

then arguing as before, we get

$$\epsilon_q(A_2) \leq \frac{1}{2} C.$$

In the same manner as before we get a subspace  $Q_2$  of finite codimension such that  $I(F_1 \oplus F_2; Q_2) > 1 - \epsilon_2$  where  $F_2 = [x_i : i \in \sigma_2]$  and a sequence  $(y_i)_{i \in \sigma_3}$  in  $Q_2$  with  $\theta(y_i)_{i \in \sigma_3} \geq \frac{1}{C}$  and  $\epsilon_q(A_3) \leq 1/2^2 C$  where  $A_3 = (z_i)$ , and

$$z_i = \begin{cases} x_i = a_i y_i, & i \in \sigma_3 \\ 0, & \text{otherwise} \end{cases}$$

Continuing, we get a sequence  $\{Q_k\}$  such that  $I(F_1 \oplus \dots \oplus F_k; Q_k) > 1 - \epsilon_k$  where  $F_k = [x_i : i \in \sigma_k]$  and sequences  $\{y_i\}_{i \in \sigma_k}$  in  $Q_k$ ,  $\theta(y_i)_{i \in \sigma_{k+1}} \geq \frac{1}{C}$  and  $\epsilon_q(A_k) \leq \frac{1}{2^{k-1}} C$  where  $A_k = (z_i)$  and

$$z_i = \begin{cases} x_i = a_i y_i, & i \in \sigma_k \\ 0, & \text{otherwise} \end{cases}$$

Finally,

$$\epsilon_q(x_i)_1^\infty \leq \sum_{k=1}^\infty \epsilon_q(A_k)$$

$$\leq \sum \frac{1}{2^{k-1}} C$$

$$\leq 2C$$

It now follows from the construction and Theorem I.17 that  $(x_i)$  is a basic sequence. We now turn our attention to the task of defining the  $(f_i)$ . Let  $(y'_i)_{i \in \sigma_k}$  be the functionals given in I.20 corresponding to the  $(y_i)_{i \in \sigma_k}$  above. We define the  $f_i$ 's as follows:

$$f_i(x_j) = \begin{cases} y'_i(x_j) = a_i \delta_{ij}, & \text{if } i \text{ and } j \text{ are not in} \\ & \text{the same } \sigma_k \\ 0, & \text{otherwise} \end{cases}$$

It is clear that the  $f_i$ 's are bounded functionals on  $[x_i : i \in \mathbb{Z}^+]$  since  $(x_i)$  is a basic sequence. Note that if

$i \in \sigma_k$ , then  $f_i = y_i'(P_{i_k} - P_{i_{k-1}})$ . Let the basis constant of  $(x_i)$  be  $M$ . Then,

$$\begin{aligned}
 \epsilon_p^*(a_i f_i) &= \sup \left( \sum_1^\infty |a_i \langle x, f_i \rangle|^p \right)^{1/p} \\
 &= \sup \left( \sum_{k=1}^\infty \sum_{i \in \sigma_k} |a_i y_i'(P_{i_k} - P_{i_{k-1}}) x|^p \right)^{1/p} \\
 &\leq 2M \sum_{k=1}^\infty \epsilon_p^*(a_i y_i)_{i \in \sigma_k} \\
 &\leq 2M \sum \frac{C}{2^{k-1}} \\
 &= 4MC .
 \end{aligned}$$

Corollary I.22 If  $E$  is a  $D_p$ -space, then  $E$  contains a subspace  $E_0$  which is a  $S_p$ -space.

Corollary I.23 If  $E$  is a Banach space and  $(a_i)$  is as in Theorem I.21, then there is a basic sequence  $(x_i)$  in  $E$  such that  $\|x_i\| = a_i$  and  $\epsilon_2(x_i) < +\infty$ . Further, there exists functionals  $(f_i)$  on  $[x_i : i \in \mathbb{Z}^+]$  such that  $f_i(x_j) = a_i \delta_{ij}$ , and  $\epsilon_2^*(a_i f_i) < +\infty$ .

Remark I.24 If in I.21 we allowed  $E$  to be an  $S_p$ -space, then the sequence of functionals  $(f_i)$  could be taken to be defined on  $E$  instead of the subspace  $[x_i : i \in \mathbb{Z}^+]$ .

Proof: In the proof of I.21, let  $P_{\sigma_k}$  be the projections from  $E$  onto  $[x_i : i \in \sigma_k]$ . Now define  $F_i = f_i P_{\sigma_k}$  where  $i \in \sigma_k$ . Now  $\epsilon_p^*(a_i F_i) = \sup_{x \in \bigcup_E} (\sum_{i=1}^{\infty} |a_i F_i(x)|^p)^{1/p}$

$$= \sup_{k=1}^{\infty} \sum_{i \in \sigma_k} (\sum_{i \in \sigma_k} |a_i F_i(x)|^p)^{1/p}$$

$$\leq \sum_{k=1}^{\infty} \sup_{i \in \sigma_k} (\sum_{i \in \sigma_k} |a_i F_i(x)|^p)^{1/p}$$

$$\leq \sum_k \sup_{i \in \sigma_k} (\sum_{i \in \sigma_k} |a_i f_i P_{\sigma_k}(x)|^p)^{1/p}$$

$$\leq \sum_k ||P_{\sigma_k}|| \sup_{\substack{||z|| \leq 1 \\ z \in [x_i]}} (\sum_{i \in \sigma_k} |a_i f_i(z)|^p)^{1/p}$$

$$\leq \sum_k ||P_{\sigma_k}|| \frac{1}{2^{k-1}} C$$

$$\leq 2MC, \text{ where } M = \sup_k ||P_{\sigma_k}||.$$

Our final result of this chapter combines the Principle of Local Reflexivity with the techniques of I.21 to construct an analogous sequence in  $E'$ . These two results will almost characterize the diagonal operators on  $\ell_p$ -spaces which are totally factorable.

**Theorem I.25** Let  $E$  be a Banach space and  $(\lambda_i)$  as in I.21. Then, there exists a basic sequence  $(x_i)$  in  $E$ , and  $(f_i)$  in  $E'$  satisfying

$$i. \quad ||x_i|| \leq 2$$

$$ii. \quad \langle x_i, f_j \rangle = \delta_{ij}$$

$$iii. \quad \varepsilon_2^*(\lambda_i f_i) < +\infty.$$

Proof: Set a subsequence  $(i_k)$  of  $Z^+$  satisfying the property: If  $i > i_k$ , then  $\lambda_i \leq \frac{1}{2^k}$ . Without loss of generality we may assume that  $\lambda_i \leq 1$  for all  $i$ . Let  $\rho_k = \{1, 2, \dots, i_k\}$  and  $\sigma_k = \{i_{k-1}+1, \dots, i_k\}$ , where  $k = 1, 2, \dots$ ,  $i_0 = 0$ , and  $m_k = i_k - i_{k-1}$ . By I.9, there exists  $[f_i: i \in \sigma_1]$  in  $E'$  such that  $d([f_i: i \in \sigma_1], \ell_2^{m_1}) < 2$  and  $\theta(f_i)_{i \in \sigma_1} \geq \frac{1}{2}$ . Let  $\{F_i: i \in \sigma_1\}$  consist of elements of  $E'$  which are biorthogonal to  $(f_i)_{i \in \sigma_1}$  with  $||F_i|| < 2$ . By local reflexivity, there exists  $(x_i)_{i \in \sigma_1}$  in  $E$  biorthogonal to  $(f_i)_{i \in \sigma_1}$  and  $||x_i|| \leq 2$ . Let  $Y_1 = {}^\perp[f_i: i \in \sigma_1]$ , then  $E = Y_1 \oplus [x_i: i \in \sigma_1]$  and  $Y_1^\perp = [f_i: i \in \sigma_1]$ . Pick a subspace  $Q_1$  of  $E'$  such that  $I([f_i: i \in \sigma_1]; Q_1) > 1 - \varepsilon_1$ . If  $q \in Q_1$ , then there exists a  $y \in Y$  such that  $q(y) \neq 0$ . But, since  $[x_i: i \in \sigma_1] \cap [y] = 0$ , then  $q([x_i: i \in \sigma_1]) \equiv 0$ , i.e.  $q \in [x_i: i \in \sigma_1]^\perp$ . Select  $(f_i)_{i \in \sigma_2}$  in  $Q_1$  such that  $d([f_i: i \in \sigma_2], \ell_2^{m_2}) < 2$  and  $\theta(f_i)_{i \in \sigma_2} \geq 1/2$ . View each  $f_i, i \in \sigma_2$ , as a linear functional on  $Y_1$ . Note that  $f_i(Y_1) \neq 0$  for each  $i \in \sigma_2$ . Now find  $(F_i)_{i \in \sigma_2}$  in  $Y_1'$  and  $(x_i)_{i \in \sigma_2}$  in  $Y_1$  as before. Now, define  $Y_2 = {}^\perp_1[f_i: i \in \sigma_2]$ . Choose  $Q_2$  to be a subspace of  $E'$  such that

$$I([f_i : i \in \sigma_2]; Q_2) > 1 - \epsilon_2 .$$

In the inductive step

$$Y_k = [f_i]_{i \in \sigma_1 \dots \sigma_k}$$

and

$$I([f_i]_{\sigma_1 \dots \sigma_k}; Q_k) > 1 - \epsilon_k .$$

Now  $(f_i)$  is a basic sequence in  $E'$  with  $\epsilon_2(\lambda_i f_i) < +\infty$  (see the proof of I.21).  $(x_i)$  is a basic sequence with coefficient functionals  $(f_i)$  by IV.5.

We end this chapter with a summary of the major results of this chapter.

**Theorem I.26** Let  $E$  be a Banach space and  $(a_i)$  be as in I.21. Then,

(i) there exists a basic sequence  $(x_i)$  and functionals  $(f_i)$  defined on  $[x_i : i \in \mathbb{Z}^+]$  such that  $\epsilon_2(x_i) < +\infty$ ,  $\epsilon_2^*(a_i f_i) < +\infty$ , and  $\langle x_i, f_j \rangle = a_i \delta_{ij}$ .

(ii) there exists a basic sequence  $(x_i)$  and functionals  $(f_i)$  defined on  $E$  such that  $\epsilon_2(x_i) < +\infty$ ,  $\langle x_i, f_j \rangle = a_i \delta_{ij}$ , and  $\|f_i\| \rightarrow 0$ .

(iii) there exists a basic sequence  $(x_i)$  in  $E$  and functionals  $(f_i)$  defined on  $E$  such that  $\|x_i\|$  is bounded,  $\langle x_i, f_j \rangle = \delta_{ij}$ , and  $\epsilon_2(a_i f_i) < +\infty$ .

We conclude this chapter with the observation that

(i) above is the best possible result, i.e. we can not get

functionals  $(f_i)$  defined on  $E$ . The operator in II.4 (a diagonal operator  $D_\alpha: \ell_2 \rightarrow \ell_2$  corresponding to an element of  $c_0$ ) shows that we can not improve I.26 such that  $f_i \in E'$  instead of  $[x_i]'$ . In II.4 we show that  $D_\infty$  factors through a subspace of  $E$ . If  $f_i \in E'$  for all  $i$ , then the factorization can be taken through  $E$ , i.e.,  $D_\infty$  would factor through every Banach space. In particular,  $D_\infty$  would factor through  $\ell_1$ . But by 0.13 this would imply that  $D_\infty$  is absolutely summing. But this is a contradiction.

## Chapter II

### TOTALLY FACTORABLE OPERATORS

In this chapter, we will use I.23 and I.25 to show that certain diagonal operators between  $\ell_p$ -spaces will factor through every Banach space. In fact we give an almost complete characterization of these operators. The only instance in which we fail is when  $p > 2$  and  $q > 2$ . We note here that this case includes the universal nuclear operators, i.e. a diagonal operator (corresponding to an  $\ell_1$ -sequence) from  $c_0$  into  $\ell_1$ . If we could prove that the nuclear operators were totally factorable, then we could in turn prove the "Little Grothendieck" conjecture: If  $K(E, E) = \eta(E, E)$ , then  $\dim E < +\infty$ .

In the first part of the chapter we introduce locally and totally factorable operators. Next, we give a series of results which almost characterize the diagonal totally factorable operators. At the end of these results we give a table listing most of our results concerning totally factorable diagonal operators. In the final chapter, we will give other examples of totally factorable operators. In particular, we will show that the natural injections

$$\ell_1 \rightarrow \ell_\infty$$

and



$$\ell_1 \rightarrow c_0$$

are totally factorable.

We end this chapter with an investigation of the Grothendieck conjecture.

To begin this chapter, we need the following notation:

For  $1 \leq p, q \leq +\infty$  and  $1/p + 1/p' = \frac{1}{q} + \frac{1}{q'} = 1$ ,

$$1. \quad \alpha(p, q) = \min \{p, p', q, q'\}$$

$$2. \quad \omega(p, q) = \max \{p, p', q, q'\}$$

$$3. \quad \phi(p, q) = \{q^{-1} - p^{-1} + \frac{1}{2}\}^{-1}$$

Note that  $1/\alpha(p, q) + \frac{1}{\omega(p, q)} = \frac{1}{\phi(p, q)} + \frac{1}{\phi(q, p)} = 1$ .

In the results to follow  $D_r: \ell_p \rightarrow \ell_q$  is a diagonal operator corresponding to a sequence  $(\lambda_i) \in \ell_r$ . For  $r = \infty$ ,  $(\lambda_i) \in c_0$ . For convenience, we will sometimes abbreviate this by  $D_r^\sim(\lambda_i)$ . Nothing is lost by assuming that  $1 > \lambda_i > 0$  for all  $i$ , so this assumption is made.

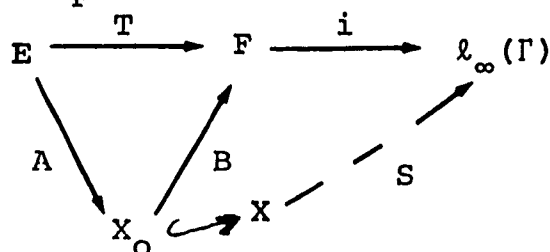
**Definition II.1** Let  $T \in L(E, F)$ .  $T$  is a locally factorable operator if for every infinite dimensional Banach space  $X$ , there exists a subspace  $X_0 \subseteq X$ , and operators  $A, B$  such that  $A \in L(E, X_0)$ ,  $B \in L(X_0, F)$  and  $T = BA$ .  $T$  is totally factorable if we can always take  $X_0$  to be  $X$  itself.

**Remark II.2** Let  $T$  be a totally (locally) factorable operator from  $E$  into  $F$  and  $V \in L(X, E)$  and  $W \in L(F, Y)$  for arbitrary

Banach spaces  $X$  and  $Y$ , then WTU is totally (locally) factorable.

**Theorem II.3** An operator  $T:E \rightarrow F$  is locally factorable if and only if  $iT:E \rightarrow \ell_\infty(\Gamma)$  is totally factorable, where  $i$  is an isometry from  $F$  into  $\ell_\infty(\Gamma)$  for some  $\Gamma$ .

**Proof:** (Sufficiency) This follows immediately from the fact that  $\ell_\infty(\Gamma)$  has the extension property. See the following diagram. Here  $\Gamma = U_F$ .



$S$  extends  $iB$ .

**Proof of Necessity:** If  $iT:E \rightarrow \ell_\infty(\Gamma)$  is totally factorable, then given  $X$ , there exists  $A \in L(E, X)$  and  $B \in L(X, \ell_\infty(\Gamma))$  such that  $T = BA$ . Let  $X_0 = B^{-1}(iF)$  and  $\hat{B} = i^{-1}B|_{X_0}$ . Then  $T = \hat{B}A$ , since  $A(E) \subseteq X_0$ .

**Proposition II.4** Let  $D_\infty: \ell_2 \rightarrow \ell_2$ , then  $D_\infty$  is locally factorable.

**Proof:** Let  $E$  be an arbitrary Banach space. Let  $D_\infty \sim (\lambda_i)$ . Without loss of generality we may assume that  $0 < \lambda_i < 1$ . For the sequence  $(\sqrt{\lambda_i})$ , let  $E_0 = [x_i: i \in \mathbb{Z}^+]$ , where  $(x_i)$  is the basic sequence from II.23 with corresponding functionals  $(f_i)$ . Define  $A: \ell_2 \rightarrow E_0$  by  $A(\xi_i) = \sum_{i=1}^{\infty} \xi_i x_i$ , then

$$\begin{aligned}
||A(\xi_i)|| &= ||\sum_1^{\infty} \xi_i x_i|| \\
&= \sup_{f \in \bigcup_E} |f(\sum \xi_i x_i)| \\
&= \sup |\sum \xi_i f(x_i)| \\
&\leq \sup ||\xi_i||_2 \varepsilon_2(x_i)
\end{aligned}$$

Hence  $||A|| \leq \varepsilon_2(x_i)$ . Now define  $B: E_0 \rightarrow \ell_2$  by  $B(e) = (\sqrt{\lambda_i} f_i(e))$ . Then  $||B|| \leq \varepsilon_2(f_i)$ . Clearly  $D_{\infty} = BA$ .

Corollary II.5 Let  $H_1$  and  $H_2$  be two Hilbert spaces and  $T \in K(H_1, H_2)$ , then  $T$  is locally factorable.

Proof: If  $T \in K(H_1, H_2)$ , then  $T = \sum \lambda_i e_i \otimes f_i$  where  $(\lambda_i) \in c_0$  and  $(e_i), (f_i)$  are orthogonal subsets of  $H_1$  and  $H_2$  respectively. The proof is now constructed as in II.4.

Lemma II.6 Let  $E$  be an arbitrary Banach space and  $H$  a Hilbert space. If  $T \in K(E, H)$ , then  $T = BA$ , where  $B \in K(H, H)$  and  $A \in L(E, H)$ . Furthermore  $T$  is locally factorable.

Proof: Since  $T$  is compact, the closure of the range of  $T$  must be separable. Hence we may assume that  $H = \ell_2$ . The proof requires use of a well-known result concerning compactness in a space with a basis: Let  $X$  be a subset of a Banach space with basis  $(x_i)$ , then the closure of  $X$  is compact if and only if  $(\sum_1^{\infty} a_i x_i)_n$  coverages uniformly to zero in norm for every  $x = \sum_1^{\infty} a_i x_i \in X$ . For  $x \in E$ ,  $Tx = \sum_1^{\infty} \langle x, T_i \rangle e_i$ .

Pick  $N_1$  such that

$$\left\| \sum_{N_1}^{\infty} \langle x, T_i \rangle e_i \right\| < \frac{1}{2^2}$$

for every  $x \in U_E$ . Let  $\sigma_k = \{N_{k-1} + 1, N_k\}$ . In general, pick  $N_m$  such that

$$\left\| \sum_{N_m}^{\infty} \langle x, T_i \rangle e_i \right\| \leq \frac{1}{2^{2m}}$$

for each  $x \in U_E$ . Then  $T = BA$  where  $A: E \rightarrow \ell_2$  and  $B: \ell_2 \rightarrow \ell_2$  are defined by formulas

$$A(x) = \sum_{m=1}^{\infty} \sum_{i \in \sigma_m} 2^{m-1} \langle x, T_i \rangle e_i$$

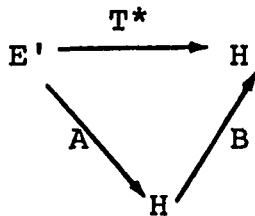
$$B(\xi_i) = \sum_{m=1}^{\infty} \sum_{i \in \sigma_m} \frac{1}{2^{m-1}} \xi_i e_i.$$

That  $T$  is locally factorable follows from II.5.

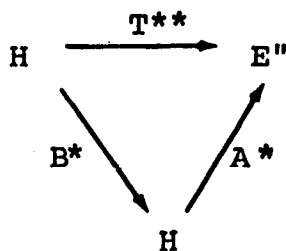
**Corollary II.7** Let  $H$  be a Hilbert space and  $E$  a Banach space, then every compact operator  $T: H \rightarrow E$  is locally factorable.

**Proof:** If  $T: H \rightarrow E$  is compact, then  $T^*: E' \rightarrow H$  is compact.

By II.6 we have the following diagram



with  $B$  compact. Thus, we have by taking adjoints again the diagram



Note that  $T^{**}x = Tx$  for every  $x \in H$ . Let  $\tilde{A} = A^*P$  where  $P$  is a projection on  $(A^*)^{-1}(E)$ . Then  $T = \tilde{A}B^*$ . Since  $B^*$  is compact,  $B^*$  and therefore  $T$  is locally factorable.

**Theorem II.8** Every quasi-nuclear  $T: E \rightarrow F$  is locally factorable.

**Proof:** If  $T$  is a quasi-nuclear operator, then by 0.10  $iT$  is nuclear, and hence factors compactly through  $\ell_2$  by Grothendieck's factorization of nuclear operators (0.7).

Let  $iT = T_2 T_1$  where  $T_1 \in K(E, \ell_2)$  and  $T_2 \in K(\ell_2, \ell_\infty(\Gamma))$ . Let  $T_1 = BA$  be a local factorization of  $T_1$  and  $\tilde{B} = i^{-1}T_2PB$  where  $P: \ell_2 \rightarrow T_2^{-1}(iF)$  is a projection. Thus  $T = \tilde{B}A$  is a local factorization of  $T$ .

Before beginning our characterizations of some of the diagonal totally factorable operators we will need results of Garling[11] and Tong[37], respectively.

**Theorem II.9[11]** The operator  $D_s$  is  $r$ -absolutely summing ( $1 \leq r < \infty$ ) from  $\ell_p$  into  $\ell_q$  if and only if the following conditions are satisfied:

- (i) if  $1 \leq p' \leq 2$  and  $p' < q$ ,  
 $s = p'$  for  $0 \leq r \leq p'$   
 $s = r$  for  $p' \leq r \leq q$   
 $s = q$  for  $q \leq r$ .
- (ii) if  $1 \leq p' = q < 2$ ,  
 $s = p'$  for  $1 \leq s \leq p'$ ,  
 $s = p'$  for  $p' \leq r$ ;
- (iii) if  $p' = q = 2$ ,  $s = 2$  for all values of  $r$ ;
- (iv) if  $1 \leq q \leq p' \leq 2$ ,  $s = q$  for all values of  $r$ ;
- (v) if  $1 \leq q \leq 2$  and  $2 < p' \leq \infty$ ,  
 $s = \phi(p, q)$  for all values of  $r$ ;
- (vi) if  $2 \leq q \leq p' < \infty$ ,  
 $s = p'$  for  $0 \leq r \leq p$
- (vii) if  $2 < p' < q \leq \infty$ ,  
 $s = p'$  for  $0 \leq r \leq p'$ ,  
 $s = r$  for  $p' \leq r \leq q$ ;

and

(viii) if  $2 \leq q \leq p' = \infty$ ,  $s = \infty$  for all values of  $r$ .

Theorem II.10[37] The diagonal operator  $D_s$  is nuclear from  $\ell_p$  into  $\ell_q$  if and only if the following conditions are satisfied:

- i. if  $1 \leq q \leq p \leq \infty$ ,  $s = 1$
- ii. if  $1 \leq p < q < \infty$ ,  $s = pq/pq + p - q$
- iii. if  $q = \infty$  and  $1 < q < \infty$ ,  $s = p/p - 1$
- iv. if  $q = \infty$  and  $p = 1$ ,  $s = \infty$

Before proceeding, we will need a result concerning elements of  $\ell_p$ .

Lemma II.11 If  $1 \leq p < \infty$ ,  $(\alpha_i) \in \ell_p$  and  $\epsilon > 0$ , then there exists a sequence of numbers of  $\lambda_i$ ,  $0 \leq \lambda_i \leq 1$ , tending to zero, such that  $\|(\alpha_i/\lambda_i)\|_p < \|(\alpha_i)\|_p + \epsilon$ .

Proof: Let  $N_{j+1} > N_j$  be such that  $\sum_{j=N_j+1}^{\infty} |\alpha_i|^p < \epsilon^p / 2^{4(j+1)p}$  for  $j = 1, 2, \dots$ . Let  $\sigma_j = \{N_j+1, \dots, N_{j+1}\}$ .

Let

$$\lambda_i = \begin{cases} 1 & \text{for } i \leq N_1 \\ \frac{1}{j} & \text{for } i \in \sigma_j \end{cases}$$

then  $\sum_{i=1}^{\infty} |\alpha_i/\lambda_i|^p$

$$= \sum_1^{N_1} |\alpha_i|^p + \sum_{j=1}^{\infty} \left( \sum_{i \in \sigma_j} j^p |\alpha_i|^p \right)$$

$$= \sum_1^{N_1} |\alpha_i|^p + \sum_{j=1}^{\infty} j^p \left( \sum_{i \in \sigma_j} |\alpha_i|^p \right)$$

$$\leq \sum_1^{N_1} |\alpha_i|^p + \sum_{j=1}^{\infty} j^p \epsilon^{p/4(j+1)p}$$

$$\leq \sum_1^{\infty} \epsilon^{p/2p(j+1)} + \sum_1^{N_1} |\alpha_i|^p$$

$$\leq (\epsilon/2)^p + \sum_1^{N_1} |\alpha_i|^p$$

which implies that

$$\| |\alpha_i/\lambda_i| \|_p \leq \epsilon/2 + \left( \sum_1^{N_1} |\alpha_i|^p \right)^{1/p}$$

and the desired result is obtained.

Corollary II.12 Let  $(\lambda_i) \in \ell_r$ , then  $\lambda_i = \alpha_i \beta_i \gamma_i$  where  $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ ,  $(\alpha_i) \in \ell_p$ ,  $(\beta_i) \in \ell_q$ , and  $(\gamma_i) \in c_0$ .

Theorem II.13 The diagonal operator

$$D_{\infty}: \ell_1 \rightarrow \ell_2$$

is totally factorable.



Proof: Let  $D_\infty \sim (\lambda_i)$ . Let  $E$  be an infinite dimensional Banach space. Then by I.25 there exists a basic sequence  $(x_i)$  in  $E$  and  $(f_i)$  in  $E'$  with  $\|x_i\| \leq 2$ ,  $\langle x_i, f_j \rangle = \delta_{ij}$  and  $\varepsilon_2(\lambda_i f_i) < +\infty$ . Define  $A: \ell_1 \rightarrow E$  by

$$A(\xi_i) = \sum_{i=1}^{\infty} \xi_i x_i;$$

and

$$B: E \rightarrow \ell_2 \text{ by}$$

$$Be = (\langle e, \lambda_i f_i \rangle).$$

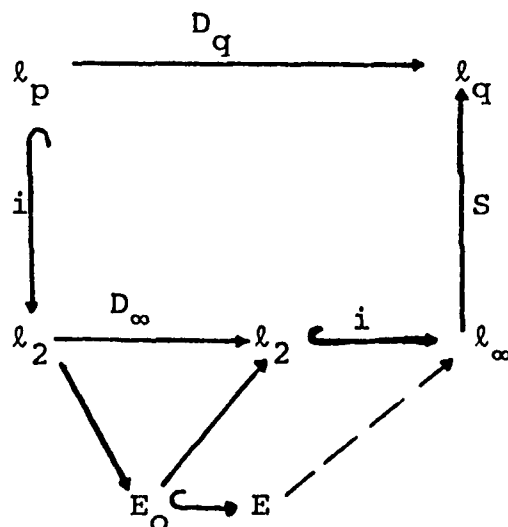
Then  $\|A\| \leq 2$ ,  $\|B\| \leq \varepsilon_2^* (\lambda_i f_i)$  and  $D_\infty = BA$ .

Theorem II.14 If  $p \leq 2$ , then

$$D_q: \ell_p \rightarrow \ell_q$$

is totally factorable.

Proof: Let  $D_q \sim (\lambda_i)$  and  $E$  be infinite dimensional. By II.11  $\lambda_i = \alpha_i \beta_i$  where  $(\alpha_i) \in c_0$  and  $(\beta_i) \in \ell_q$ . Now, consider the following diagram



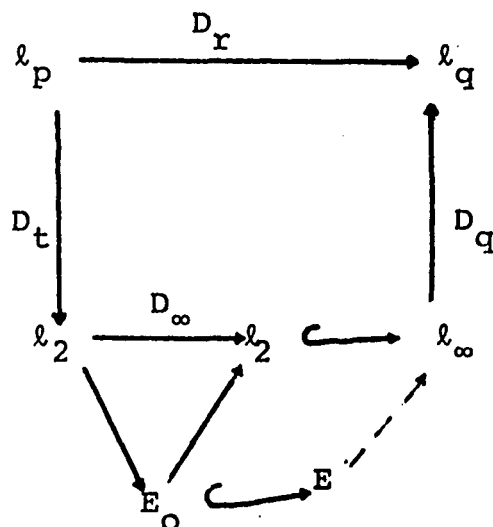
where  $i$  and  $j$  are natural injections;  $D_\infty \sim (\alpha_i)$  and  $S$  is a diagonal operator corresponding to  $(\beta_i)$ . By II.4  $D_\infty$  factors through a subspace  $E_0$  of  $E$ . Let  $D_\infty = BA$  be the factorization through  $E_0$ . Then  $jB$  maps  $E_0$  into  $l_\infty$ . Since  $l_\infty$  has the extension property,  $jB$  can be extended to an operator  $j\hat{B}$  from  $E$  into  $l_\infty$ . Hence  $D_q$  factors through  $E$ .

Theorem II.15 If  $p \leq 2$ , then

$$D_r: l_p \rightarrow l_q$$

is totally factorable when  $r = \phi(p, q)$ .

Proof: Let  $D_r \sim (\lambda_i)$  and  $\lambda_i = \alpha_i \beta_i \gamma_i$  (by II.12) where  $\frac{1}{t} = \frac{1}{2} - \frac{1}{p}$ ,  $(\lambda_i) \in \ell_t$ ,  $(\alpha_i) \in \ell_q$ , and  $(\beta_i) \in c_0$ . Now consider the diagram



where

$$D_t \sim (\alpha_i)$$

$$D_q \sim (\beta_i)$$

$$D_\infty \sim (\gamma_i)$$

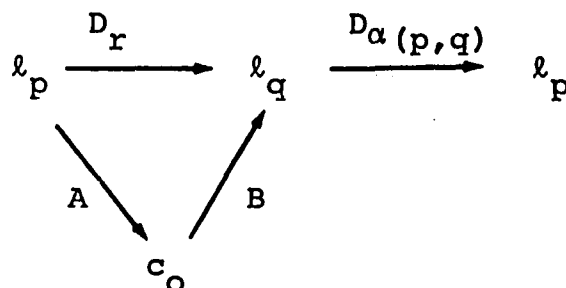
The proof is now completed as in II.14.

Theorem II.16 If  $1 \leq p \leq 2 \leq q$ , then

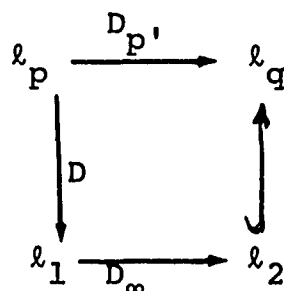
$$D_r: l_p \rightarrow l_q$$

is totally factorable if and only if  $r = \omega(p, q)$ .

**Proof:** Consider the following diagram



Now,  $D_{\alpha(p,q)}$  is absolutely summing by (i), (ii), (iii), and (iv) of II.9. The operator  $D_{\alpha(p,q)}^B$  is absolutely summing by 0.16, and hence  $D_{\alpha(p,q)}^B$  and  $D_{\alpha(p,q)}^{D_r} = D_{\alpha(p,q)}^{BA}$  is nuclear by 0.15 and 0.16. By II.10  $D_{\alpha(p,q)}^{D_r}$  is a diagonal operator corresponding to an  $\ell_1$  sequence. Hence  $r \leq \omega(p,q)$ . Sufficiency: If  $q \leq p'$ , then  $\omega(p,q) = p'$  and  $D_{p'}: \ell_p \rightarrow \ell_q$  factors in the following manner.



$D$  is a diagonal operator corresponding to an element of  $\ell_{p'}$ . The map  $D_{p'}$  is totally factorable since  $D_\infty$  is totally factorable. If  $p' \leq q$  then  $D_q: \ell_p \rightarrow \ell_q$  is totally factorable by II.14.

**Theorem II.17** If  $1 \leq q \leq 2 \leq p$ , and if

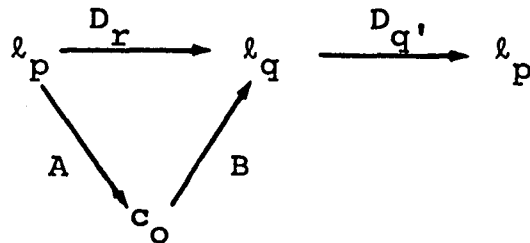
$$D_r: \ell_p \rightarrow \ell_q$$

is totally factorable then  $r \leq \alpha(p, q)$ .

Proof: We need to break the proof into two parts.

Part 1.  $q \leq p'$

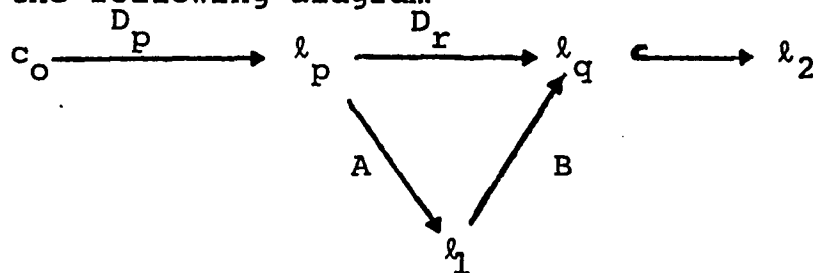
Consider the following diagram



By II.9, parts (iii), (vi) and (vii)  $D_{q'}$  is absolutely summing. The proof is now completed as in II.16.

Part 2.  $p' \leq q$

Consider the following diagram



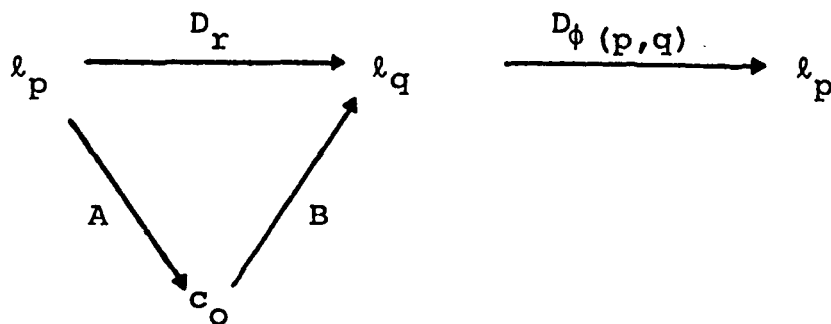
By 0.13 jB is absolutely summing. Now as before, the proof is completed.

Theorem II.18 If  $p, q \leq 2$ , then

$$D_r: \ell_p \rightarrow \ell_q$$

is totally factorable if and only if  $r = \phi(p, q)$ .

Proof: (Necessity) Consider the following diagram



$D_\phi(q,p)$  is absolutely summing by II.9 (vi). Now, as in II.16 the proof is completed. Sufficiency: For  $r = \phi(p,q)$ , the operator  $D_r$  factors in the following manner.

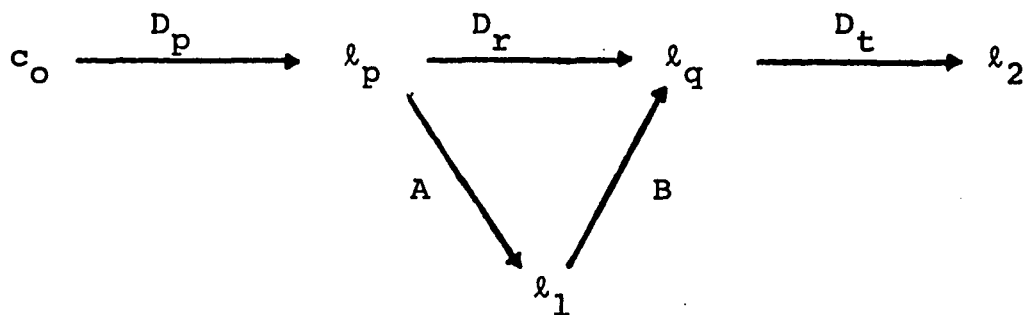
$$l_p \xrightarrow{D_p} l_1 \xrightarrow{D_\infty} l_2 \xrightarrow{D_t} l_q$$
 with  $D_\infty$  totally factorable and  $\frac{1}{t} = \frac{1}{q} - \frac{1}{2}$ .

Theorem II.19 If  $p, q \leq 2$ , then

$$D_r: l_p \rightarrow l_q$$

is totally factorable if and only if  $r = \phi(p,q)$ .

Proof: (Necessity) Consider the following diagram



where  $\frac{1}{t} = \frac{1}{2} - \frac{1}{q}$ . The proof is completed as in II.17.

Sufficiency: See II.15.

We take this opportunity to summarize the preceeding results in table form:

$D_r: \ell_p \rightarrow \ell_q$	Sufficient Condition $r =$	Necessary Condition $r =$
$p, q \leq 2$	$\phi(p, q)$	$\phi(p, q)$
$p, q \geq 2$	$\phi(p, q)$	$\phi(p, q)$
$p \leq 2 \leq q$	$\omega(p, q)$	$\omega(p, q)$
$p \geq 2 \geq q$	$\phi(p, q)$	$\alpha(p, q)$

Remark II.20 Combining the proofs of propositions II.15 and II.19, we obtain new information about the Hilbert-Schmidt operators.

Definition II.21 Let  $E$  and  $F$  be Hilbert spaces, then  $T$  is a Hilbert-Schmidt operator if for every pair of complete orthonormal systems  $(e_i)_{i \in J}$ ,  $(f_j)_{j \in J}$  of  $E$  and  $F$  respectively, we have

$$\sum_I ||Te_i||^2 = \sum_J ||T^*f_j||^2 < +\infty.$$

Theorem II.22  $T$  is a Hilbert-Schmidt operator if and only if  $T$  is a totally factorable operator.

Corollary II.23 (Pietsch [34]) If  $T$  is a Hilbert-Schmidt operator, then  $T$  is absolutely summing.

Proof: Combine 0.13 and II.22

Remark II.23 A. Pelczyński has shown in [30] that the Hilbert-Schmidt operators coincide with the absolutely  $p$ -summing operators for all  $p$ .

Remark II.24 It was previously pointed out in Chapter I that it is conjectured that every Banach space is either a  $S_1$ -,  $S_2$ -, or  $S_\infty$ -space. If this conjecture is true, then we could show in the case  $p \geq 2 \geq q$  that the operator  $D_{\alpha}(p, q)$  factors through every Banach space.

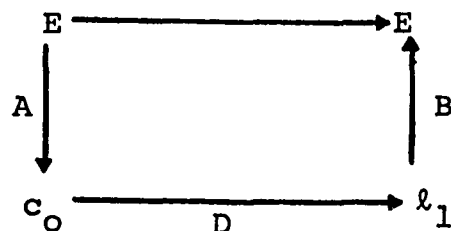
In the rest of this chapter we investigate the two conjectures of Grothendieck discussed previously. We give a proof of the result of Johnson[18] which answers the "Little Grothendieck" conjecture for the case  $E$  has LUST. In fact, we prove the result in the more general case where  $E$  is a  $S_p$ -space for any  $p$ . We achieve this by showing that if every nuclear operator factors through  $E$ , then  $K(E, E) \neq \eta(E, E)$ . This naturally brings up the question: What Banach spaces have the property that they admit a factorization of every nuclear operator?

We end this chapter with another interesting question: What conditions can be placed on the domain and range of a nuclear operator  $T: E \rightarrow F$  to insure that it is totally factorable? One answer is: If  $E$  is a  $L_2$ -space (Hilbert space) or  $F$  is a  $L_p$ -space for  $p \geq 2$ , then  $T$  is totally factorable. We conjecture that if  $E$  is a  $L_p$  for  $p \leq 2$ , then  $T$  is totally factorable.

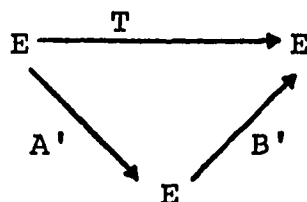


**Theorem II.25** Let  $E$  be an infinite dimensional Banach space and suppose that every nuclear operator factors through  $E$ , then  $K(E,E) \neq \mathfrak{n}(E,E)$ .

**Proof:** Let  $T$  be a nuclear operator on  $E$ , then by 0.7, the following factorization is possible



where  $A, B$  are compact and  $D$  is nuclear. Now, if every nuclear operator factors through  $E$ , then  $D$  factors through  $E$  producing the following diagram



where  $A', B'$  are compact, and hence nuclear by assumption.

In a paper to be published by Retherford, Johnson and König, they prove the following result: If  $T=BA$  where  $B \in \Pi_2(E,E)$  and  $A \in \Pi_2(E,E)$  then  $\sum |\lambda_i(T)| < +\infty$ , where  $(\lambda_i(T))$  are the eigenvalues of  $T$ . They also prove that if  $T \in \mathfrak{n}(E,E)$  implies  $\sum |\lambda_i(T)| < +\infty$ , then  $E$  is isomorphic to a Hilbert space. This contradicts the diagram since it is well-known that there exists compact operators on infinite dimensional Hilbert spaces which are not nuclear.

The above results suggest that the problem of showing that every nuclear operator is totally factorable must be

very difficult. (In fact, all we really need to solve the "Little Grothendieck" conjecture is to show that every nuclear operator on  $E$  will factor compactly through  $E$ ). The next result gives equivalent conditions for a nuclear operator to factor through a Banach space  $E$ .

**Theorem II.26** Let  $E$  be a Banach space, then the following statements are equivalent:

- (i) Every nuclear operator factors through  $E$ .
- (ii) Every nuclear diagonal operator  $D_1: c_0 \rightarrow \ell_1$  factors through  $E$ .
- (iii) For every  $(\lambda_i) \in \ell_1$ , there exists sequences  $(x_i)$  and  $(f_i)$  in  $E$  and  $E'$  respectively such that  $\varepsilon_1(x_i) < +\infty$ ,  $\varepsilon_1^*(f_i) < +\infty$ , and  $\langle x_i, f_j \rangle = \lambda_i \delta_{ij}$ .

**Proof:** (i) and (ii) are equivalent by 0.7. (ii) implies (iii). Let  $D_1 = BA$  be a factorization of  $D_1$  through  $E$ . Define  $x_i = Ae_i$ . Since  $\varepsilon_1(e_i) < +\infty$  it follows that  $\varepsilon_1(Ae_i) = \varepsilon_1(x_i) < +\infty$ . Define  $(f_i)$  to be the coordinate functional obtained from the operator  $B: E \rightarrow \ell_1$ . By definition  $\varepsilon_1^*(f_i) < +\infty$ .

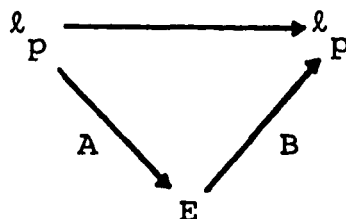
(iii) implies (ii). Define  $A: c_0 \rightarrow E$  by  $A(\xi_i) = \sum \xi_i x_i$  and  $B: E \rightarrow \ell_1$  by  $Be = \langle e, f_i \rangle$ .

In the results to follow, we essentially show (although, not directly) that every  $S_p$ -space satisfies II.26 iii. And, since every space with LUST is either a  $S_1$ -,  $S_2$ - or  $S_\infty$ -space, it follows that every nuclear operator factors through every

space with LUST.

**Theorem II.27** Let  $E$  be a  $S_p$ -space and  $D_\infty: \ell_p \rightarrow \ell_p$  a diagonal operator, then  $D_\infty$  factors through  $E$ .

**Proof:** Let  $D_\infty \sim (\lambda_i) \in c_0$ . Apply I.24 to the sequence  $(\sqrt{\lambda_i})$  to obtain a basic sequence  $(x_i)$  in  $E$  and functionals  $(f_i)$  defined on  $E$  satisfying  $\varepsilon_q(x_i) < +\infty$ ;  $\|x_i\| = \sqrt{\lambda_i}$ ;  $f_j(x_i) = \sqrt{\lambda_i} \delta_{ij}$  and  $\varepsilon_p^*(\sqrt{\lambda_i} f_i) < +\infty$ . Consider the following diagram



where  $A(x_i) = \sum \xi_i x_i$  and  $Bx = (\sqrt{\lambda_i} f_i(x))$ .

Then  $\|A\| \leq \varepsilon_p(x_i)$  and  $\|B\| \leq \varepsilon_p^*(\sqrt{\lambda_i} f_i)$ . Clearly  $D_\infty = BA$ .

**Theorem II.28** Every nuclear operator will factor compactly through any  $S_p$ -space. In particular, every nuclear operator will factor compactly through every space with LUST.

**Proof:** By II.26 we may assume that our nuclear operator is a diagonal operator,  $D_1: c_0 \rightarrow \ell_1$  corresponding to an element of  $\ell_1$ . By II.2 a  $D_1 = D_p D_\infty D'_p$  where  $D_p: c_0 \rightarrow \ell_p$ ,  $D_\infty: \ell_p \rightarrow \ell_p$  and  $D'_p: \ell_p \rightarrow \ell_p$ . Now  $D_p$  and  $D'_p$  are compact, and by II.27  $D$  factors through every  $S_p$ -space.

**Corollary II.29** Let  $E$  be a  $S_p$ -space for some  $p$ , then  $K(E, E) \neq n(E, E)$ .

Proof: See II.25 and II.28

Corollary II.30 If  $E$  is a  $S_p$  - space, then for any positive integer  $m$ , there exists a compact operator  $T$  on  $E$  such that  $T^m$  is not nuclear, but  $T^{m+1}$  is nuclear. Also, there exists a compact operator  $S$  on  $E$  such that  $S^n$  is not nuclear for any  $n$ .

Proof: See II.10 and II.27.

We conclude this chapter by showing that certain conditions on the domain and range insures that a nuclear operator will be totally factorable.

Theorem II.31 Let  $H$  be a Hilbert space; and  $T:H \rightarrow F$  a nuclear operator, then  $T$  is totally factorable.

Proof: By 0.7 the following factorization of  $T$  is possible.

$$\begin{array}{ccc}
 H & \xrightarrow{\quad} & F \\
 \downarrow A & & \uparrow B \\
 \ell_\infty & \xrightarrow{\quad D \quad} & \ell_1
 \end{array}$$

where  $A$  is compact. By II.8  $A$  is locally factorable. But since  $\ell_\infty$  has the extension property,  $A$  is also totally factorable.

Our next goal is to show that a nuclear operator into a  $L_p$ -space for  $p \geq 2$  must also be totally factorable. To aid us in this task, we need two major results, due to

Kwapien[20] and Gordon and Lewis[12].

Theorem II.32 Let  $F$  be an  $L_r$ -space where  $1 < r \leq 2$  and  $1 < p \leq 2$ , then  $\pi_p(F, E) = \pi_1(F, E)$ .

Theorem II.33 If  $E$  has LUST and  $T \in \pi_1(E, F)$ , the  $T$  factors through some  $L_1(\mu)$ -space.

Theorem II.34 Let  $F$  be a  $L_p$ -space for  $p \geq 2$  and  $T: E \rightarrow F$  a nuclear operator, then  $T$  is totally factorable.

Proof: Case  $p < +\infty$ . Because of 0.7 we may assume that our situation is as follows

$$c_0 \xrightarrow{D_1} \ell_1 \xrightarrow{K} F$$

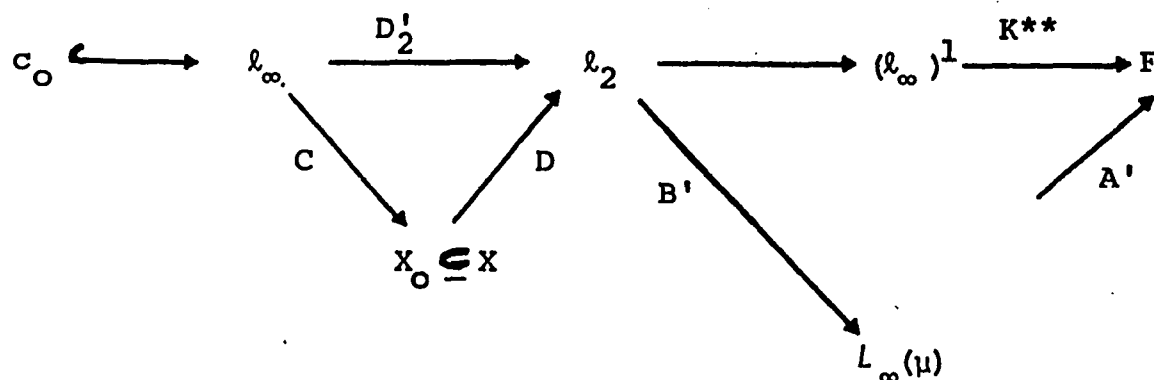
Now  $D_1$  factors through  $\ell_2$  giving us the diagram

$$\begin{array}{ccccc} c_0 & \xrightarrow{D_1} & \ell_1 & \xrightarrow{K} & F \\ & \searrow D_2 & \nearrow D'_2 & & \\ & \ell_2 & & & \end{array}$$

Taking adjoints our diagram becomes

$$F' \xrightarrow{K^*} \ell_\infty \xrightarrow{D'_2} \ell_2 \xrightarrow{D_2} \ell_1$$

By II.9  $D'_2: \ell_\infty \rightarrow \ell_2$  is absolutely 2-summing. By II.32  $D'_2 K^*$  is absolutely 2-summing. Now by II.33  $D'_2 K^*$  factors through some  $L_1(\mu)$ -space. Taking adjoints again our diagram becomes



Lindenstrauss and Pelczyński[26] has shown that every  $L_p$ -space,  $1 < p < +\infty$  is reflexive. Also, it is well known that the dual of  $L_1(\mu)$  is  $L_\infty(\mu)$ .  $B'D$  is extendable since its range lies in a  $L_\infty(\mu)$ -space, and  $T = KD_2D'_2j$ .

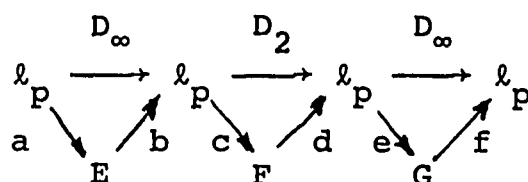
Case  $p = +\infty$ . Since  $D'_2$  is compact it factors compactly through a subspace  $X_0$  of  $X$ . Since an  $L_\infty$ -space has the extension property of compact operators[27],  $B'D$  can be extended to all of  $X$ .

### III. PRODUCTS

In this chapter, we investigate the relationship between the "Big Grothendieck" conjecture and totally factorable operators. In the beginning we use results and techniques of Chapter II in order to obtain results of Johnson [18] and Retherford and Stegall [36]. Next, we use Banach Ideal theory to show that the product of two totally factorable operators is Cohen 2-nuclear. We end the chapter by showing that the product of two absolutely 2-summing operators is totally factorable.

Theorem III.1. If  $E$  and  $G$  are  $S_p$ -space, then there exists non-nuclear operators in  $L(E, F)$  and  $L(F, G)$ .

Proof: Consider two diagonal operators  $D_2$  and  $D_\infty$  on  $\ell_p$  such that  $D_2 D_\infty$  does not correspond to an  $\ell_1$  sequence. Consider the diagram



The factorizations  $D_\infty = ba = fe$  is possible by II.26. And the factorization  $D_2 = dc$  is possible by II.18 and II.19. By II.10 neither  $D_2 D_\infty$  nor  $D_\infty D_2$  is nuclear, hence  $cb$  and  $ed$  are not nuclear.

Corollary III.2. If either  $E$  or  $G$  is an  $S_p$ -space, then there exists a non-nuclear operator from  $E$  into  $G$ .

As an immediate corollary to the proof of III.1, we obtain Retherford and Stegall's fully nuclear version of the "Big Grothendieck" conjecture [36].

Definition III.3. An operator  $T:E \rightarrow F$  is said to be fully nuclear if the astriction operator  $T_a:E \rightarrow \overline{T(E)}$  is nuclear. The set of fully nuclear operators from  $E$  into  $F$  is denoted by  $F_n(E, F)$ .

Corollary III.4. Let  $F$  and  $G$  be Banach spaces, then there exists a non-fully nuclear operator from  $F$  into  $G$ .

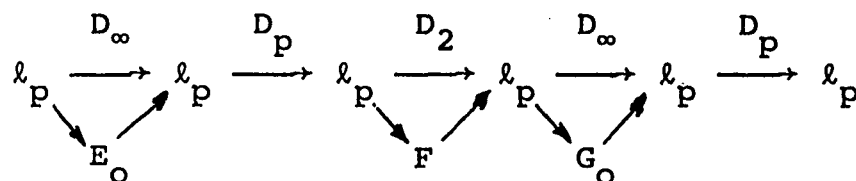
Proof: In the proof of III.1 take  $p = 2$ . Then by II.4  $D_\infty = fe$  can be taken to be a locally factorization through a subspace  $G_0 \subseteq G$ . Hence  $ed:F \rightarrow G_0$  is not nuclear and therefore  $ed:F \rightarrow G$  is not fully nuclear.

Our next result shows that for  $p > 2$  we can weaken the hypothesis of III.1 to  $D_p$ -spaces.

Theorem III.5. If  $E$  and  $G$  are  $D_p$ -spaces,  $p > 2$ , then there exists non-nuclear operators in  $L(E, F)$  and  $L(F, G)$ .

Proof: Select  $D_2$ ,  $D_p$ , and  $D_\infty$  to be diagonal operators on  $\ell_p$  such that  $D_2 D_p D_\infty$  does not correspond to an element in  $\ell_1$ . By I.22  $E$  and  $G$  contain subspaces  $E_0$  and  $G_0$  respectively which are  $S_p$ -space. Now as in III.1, the following diagram is possible.





But, by 0.19 and II.9  $D_p b$  and  $D_p f$  are extendable. Hence we obtain factorizations through  $E$  and  $G$ . The argument is now completed as in III.1.

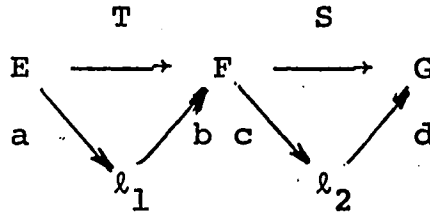
In the proof of III.1 we saw that the product of a totally factorable operator and a locally factorable operator need not be nuclear. A natural question at this stage: Is the composition of two totally factorable operators always nuclear? It is clear that a negative answer to this question solves the "Big Grothendieck" conjecture. Unfortunately, we have only been able to give answer to this question in a few special cases. Our best result concerning the general question is the following: If  $T$  and  $S$  are totally factorable, then  $ST$  is Cohen 2-nuclear. The definition of a Cohen  $p$ -nuclear operator is given in III.9.9.

**Theorem III.6.** Let  $T \in L(E, F)$  and  $S \in L(F, G)$  be totally factorable operators, then if either of the following is true:

- i.  $E$  is a  $L_\infty$ -space
- ii.  $F$  is a  $L_1$ -space
- iii.  $G$  is a  $L_2$ -space

Then  $ST$  is nuclear.

Proof: We will only prove the case when  $E$  is a  $L_\infty$ -space. The proof in the other cases is essentially the same. Consider the following diagram.



$a$  is absolutely 2-summing by 0.14.  $cb$  is absolutely 2-summing by 0.4 and 0.13. Therefore,  $ST = dcba$  is nuclear by 0.8 and 0.16.

Before investigating the general problem, we need some preliminary information.

Definition III.7. Let  $L$  denote the class of all bounded linear operators between arbitrary Banach spaces and

$L(E, F)$ , the set of all such operators between specific Banach spaces  $E$  and  $F$ . A class  $A$  of bounded linear operators is an ideal if for each set  $A(E, F) = A \cap L(E, F)$  one has

- i. if  $x' \in E'$ ,  $y \in F$ , then  $x' \otimes y \in A(E, F)$ .
- ii.  $A(E, F)$  is a linear subset of  $L(E, F)$  for each  $E$  and  $F$ , and
- iii. if  $U \in L(X, E)$ ,  $T \in A(E, F)$ , and  $V \in L(F, Y)$ , then  $VTU \in A(X, Y)$ .

A function  $\alpha: A \rightarrow \{\text{non-negative reals}\}$  is an ideal norm if one has

- iv. if  $x' \in E'$ ,  $y \in F$ , then  $\alpha(x' \otimes y) = \|x'\| \|y\|$ .
- v. if  $S, T \in A(E, F)$ , then  $\alpha(S+T) \leq \alpha(S) + \alpha(T)$ ; and
- vi. if  $U \in L(X, E)$ ,  $T \in A(E, F)$  and  $V \in L(F, Y)$ , then  $\alpha(VTU) \leq \|V\| \alpha(T) \|U\|$ .

An ideal  $A$  with norm  $\alpha$ , denoted by  $[A, \alpha]$  is called a Banach ideal if each component  $A(E, F)$  is a Banach space under the norm  $\alpha$ .

To any normed ideal  $[A, \alpha]$ , we associate the following Banach ideals:

Definition III.8.a. The adjoint ideal  $[A^*, \alpha^*]$  of the ideal  $[A, \alpha]$  is the class of all operators  $T \in (E, G)$  for which there is a  $\rho > 0$  such that for arbitrary finite dimensional Banach spaces  $E_0$  and  $G_0$  we have

$$|\text{trace } WBTX| \leq \rho \alpha(W) \|B\| \|X\|$$

for  $W \in A(G_0, E_0)$ ;  $X \in F(E_0, E)$ ; and  $B \in F(F, G_0)$ . Finally,  $\alpha^*(T) = \inf \rho$ .

b. The conjugate ideal  $[A^\Delta, \alpha^\Delta]$  is the class of all  $T \in L(E, G)$  for which there is a  $\rho > 0$  such that for all  $S \in F(G, E)$ ,

$$|\text{trace } ST| \leq \rho \alpha(S).$$

In this case,  $\alpha^\Delta(T) = \inf \rho$ .

In what is to follow we need the following Banach ideals which were introduced and studied by J. Cohen[3], S. Kwapien[24], and A. Persson and A. Pietsch[32].

Definition III.9.a. Let  $[J_p, j_p]$  denote the ideal of Cohen  $p$ -nuclear operators:  $T \in J_p(E, F)$  if there is a  $\rho > 0$  such that

$$\sigma_p(\{Tx_i\}) \leq \rho \alpha_p(\{x_i\})$$

for all finite sets  $\{x_1, \dots, x_n\}$  in  $E$ . Here  $j_p(T) = \inf \rho$ .

b. Let  $[\Gamma_p, \gamma_p]$  denote the ideal of operators factoring through  $L_p$ :  $T \in \Gamma_p(E, F)$  if for some  $L_p(\mu)$ ,  $\mu$  a positive measure, there are operators  $A \in L(E, L_p(\mu))$ ,  $B \in L(L_p(\mu), F'')$  such that  $iT = BA$ , where  $i$  is the canonical injection of  $F$  into  $F''$ . Here  $\gamma_p(T) = \inf \|A\| \|B\|$ .

c. Let  $[I_p, i_p]$  denote the ideal of  $p$ -integral operators:  $T \in I_p(E, F)$  if there is a probability measure space  $(\Omega, \mu)$  and operators  $V \in L(E, L_\infty(\mu))$  and  $W \in L(L_p(\mu), F'')$  such that  $W_j V = iT$ , where  $j$  is the canonical injection of  $L_\infty(\mu)$  into  $L_p(\mu)$  and  $i$  is the canonical injection of  $F$  into  $F''$ . Here  $i_p(T) = \inf \|V\| \|W\|$ .

Remark III.10. Before moving further, we should recall that the class  $[\Pi_p, \pi_p]$ ,  $[N, \nu]$ , and  $[K, \|\cdot\|]$  are all Banach ideals. In fact  $[N, \nu]$  is contained in every Banach Ideal.

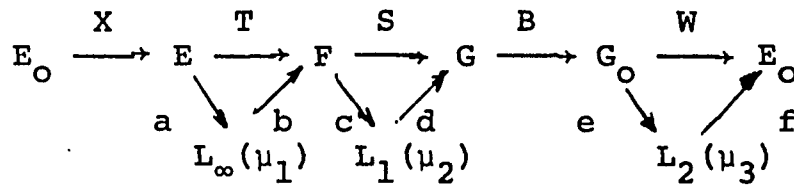
Remark III.11. Using proposition 16 of [34] and the definitions of  $[\Pi_2, \pi_2]$  and  $[I_2, i_2]$ , it is easily seen that they coincide.

Theorem III.12. [35, Kwapien] Let  $1 \leq p \leq \infty$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then  $[\Gamma_p^*, \gamma_p^*] = [J_{p'}, j_{p'}]$ .

Theorem III.13. [13, Theorem 25(a)]. Let  $T \in L(E, F)$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , then  $\pi_p^\Delta = i_{p'}$ .

Theorem III.14. Let  $T \in L(E, F)$  and  $S \in L(F, G)$  be totally factorable, then  $ST \in [J_2, j_2]$ .

Proof: The idea of this proof is to use Theorem 3.7 and show that  $ST \in [\Gamma_2^*, \gamma_2^*]$ . Consider the following diagram with  $E_0$ ,  $G_0$ ,  $X$ ,  $B$ , and  $S$  as given in Definition III.8.



The first inequality results from III.8.b, III.13, and III.11

$$\begin{aligned}
 & |\text{Trace WBSTX}| \\
 &= |\text{Trace}(feBd)(cbaX)| \\
 &\leq \pi_2(cbaX) \pi_2(feBd) \\
 &\leq \|a\| \|X\| \|f\| \pi_2(cb) \pi_1(eBd) \\
 &\leq K \|a\| \|X\| \|f\| \|cb\| \|eBd\| \\
 &\leq K \|a\| \|b\| \|c\| \|d\| \|e\| \|f\| \|B\| \|X\| \\
 &\leq K(i_\infty(T)+1)(i_1(S)+1)(i_2(W)+\epsilon) \|B\| \|X\|
 \end{aligned}$$

for arbitrary  $\epsilon > 0$ . Hence for  $\rho = K(i_\infty(T)+1)(i_1(S)+1)$ , we obtain the desired result. Note that the last inequality

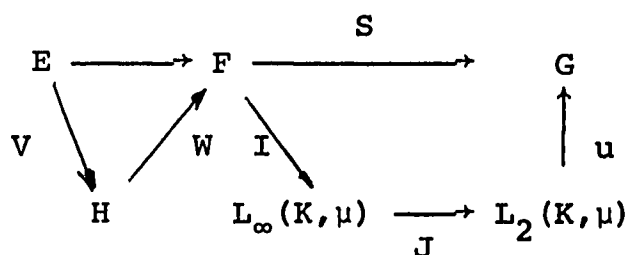
is obtained by choosing  $a, b, c, d, e, f$  in the proper fashion.

Our final result of this chapter gives a sufficient condition for the composition of two operators to be totally factorable.

**Definition III.15.** An operator  $T$  is called Hilbertian if it can be factored through a Hilbert space  $H$ .

**Theorem III.16.** If  $T$  is Hilbertian and  $S$  is absolutely 2-summing, then  $ST$  is totally factorable.

**Proof:** By 0.5 and III.15 the following factorization is possible.



$J$  is absolutely 2-summing by 0.14. Hence  $JIW$  is a Hilbert-Schmidt operator (absolutely 2-summing) and therefore totally factorable by II.22.

In particular, the composition of two absolutely 2-summing operators is totally factorable.

As we shall see in the next chapter, not every totally factorable is compact. But, the composition of two totally factorable operators is easily shown to be compact (by

first factoring through  $\ell_2$  and then through  $\ell_1$  and using the fact that every operator from  $\ell_2$  into  $\ell_1$  is compact). Furthermore, the composition of three totally factorable operators is nuclear. (Factor through  $\ell_\infty$ ,  $\ell_1$ , and  $\ell_2$ , obtaining an operator which is the composition of two absolutely 2-summing operators which must be nuclear.)

#### IV. EXAMPLES AND REMARKS

All examples of totally factorable thus far have been compact. In this chapter we give two examples of non-compact totally factorable operators. Our two examples are the natural injections of  $\ell_1$  into  $c_0$  and  $\ell_\infty$ . Although the  $\ell_\infty$  case follows immediately from the  $c_0$  case, we give a separate easier proof. At this stage (including the first two examples of this chapter), all explicitly stated examples of totally factorable operators have been diagonal operators. Although, results of chapters II and III clearly show how to obtain non-diagonal totally factorable operators. Our final example of a totally factorable operator is significant, not only because it is not a diagonal operator, but because its verification is completely different from any other we have seen.

We briefly discuss whether the sum of two totally factorable is again totally factorable.

Theorem IV.1. The natural injection  $j: \ell_1 \rightarrow \ell_\infty$  is totally factorable.

Proof: Let  $E$  be an arbitrary Banach space, then by 0.1 there is a basic sequence  $(x_i)$  in  $E$  with coefficient functionals  $(f_i)$  such that  $\|x_i\| = \|f_i\| = 1$  for every  $i$ .



Let  $(F_i)$  be the norm preserving extension of  $(f_i)$  to  $E$ .

Now define  $A: \ell_1 \rightarrow E$  and  $B: E \rightarrow \ell_\infty$  by the following formulas,

$$A(\xi_i) = \sum_{i=1}^{\infty} \xi_i x_i \text{ and}$$

$$B(e) = \langle e, f_i \rangle.$$

Clearly,  $j = BA$ .

A natural question at this stage: Is the natural injection  $\ell_1 \rightarrow c_0$  totally factorable. Although the answer to this question is yes, there seems to be no way to modify the argument above to achieve this result. A completely different proof is required. It utilizes a new result of Josefson [22] and a result easily obtainable from the remarks following theorem III.1 in [19].

**Theorem IV.2. (Josefson)** Let  $E$  be a Banach space, then there exists  $\phi_n \in E'$ ,  $n \in \mathbb{Z}^+$ , such that  $\|\phi_n\| = 1$  and  $\lim_{n \rightarrow \infty} \phi_n(z) = 0$  for all  $z \in E$ .

**Theorem IV.3. (Johnson and Rosenthal)** Let  $E$  be a Banach space and  $(\phi_n) \subset E'$  with zero the only weak\* cluster point of  $(\phi_n)$  and  $0 < \limsup \|\phi_j\| < \infty$ , then  $(\phi_n)$  has a basic subsequence  $(\phi_{n_j})$  such that if  $(f_j)$  is a sequence biorthogonal to  $(\phi_{n_j})$  in  $[\phi_{n_j}]'$  and  $T: E \rightarrow [\phi_{n_j}]'$  is defined by  $(Tx)(y) = y(x)$  for all  $y \in [\phi_{n_j}]$  and  $x \in E$ , then

$$T(E) \supset [f_j]$$

Before obtaining our desired conclusion we need the following well-known facts.

Theorem IV.4. Let  $E$  and  $F$  be Banach spaces and  $T:E \rightarrow F$ , a surjection. Then, there exists some constant  $m > 0$  such that each  $y \in F$  is of the form  $y = Tx$ , where  $\|x\| \leq m\|Tx\|$ .

Theorem IV.5. Let  $(e_i)$  be a basis for a Banach space  $E$ . Then the coefficient functionals  $(\beta_i)$  are a basic sequence.

Theorem IV.6. Let  $E$  be a Banach space then there exists a basic sequence in  $E$ ,  $(x_n)$ , with coefficient functionals  $(\psi_n)$  such that  $\{\|x_n\|\}$  is a bounded set and zero is the  $w^*$ -limit of  $(\psi_n)$ .

Proof: By IV.2 then exists in  $E'$  a sequence  $(\phi_n)$  with  $\|\phi_n\| = 1$  and  $\lim_n \phi_n(x) = 0$  for every  $x \in E$ . By IV.3 there exists a subsequence  $(\phi_{n_j})$  of  $(\phi_n)$  which is basic. Define  $\psi_j = \phi_{n_j}$ . Let  $(f_j)$  be the coefficient functionals in IV.3 and  $T$  as defined in IV.3. Let  $E_0 = T^{-1}([f_n])$ , then  $E_0$  is a Banach space. Now  $T|_{E_0} : E_0 \rightarrow [f_n]$  satisfies IV.4, hence there exists in  $E_0$  a sequence  $(x_i)$  such that  $Tx_i = f_i$  and  $\|x_n\|$  is bounded by  $2mK$ ; where  $m$  is obtained from IV.4 and  $K$  is the basis constant of  $(\psi_n)$ . Observe that  $E \subseteq [\psi_n]'$  and  $(\psi_n, x_n)$  satisfy IV.5, hence  $(x_n)$  is a basic sequence with coefficient functionals  $(\psi_n)$ .

Corollary IV.7. The natural injection  $\ell_1 \rightarrow c_0$  is totally factorable.

We now consider questions about the dual of a totally factorable operator.

Theorem IV.8. Let  $T:E \rightarrow F$  be a totally factorable operator, then  $T^*:F' \rightarrow E'$  factors through every conjugate space.

Proof: Let  $T = BA$  be a factorization through a Banach space  $X$ , then  $T^* = A^*B^*$  is a factorization through  $X'$ .

Corollary IV.9. If  $T:E \rightarrow F$  and  $T^*$  is totally factorable, then  $i_T$  factors through every conjugate space where  $i$  is the natural injection of  $F$  into  $F''$ .

Unfortunately, many unanswered questions are left by the two results. Some very obvious questions are:

1. Is the dual of a totally factorable operator also totally factorable?
2. If the dual of an operator is totally factorable, is the operator totally factorable?
3. If the dual of an operator is totally factorable, does the operator factor through every conjugate space?

We conjecture that the answer to the above questions is no. Our belief in these answers are motivated by the techniques (which appear to be necessary) used to prove the

total factorability of our next example. One final result, due to Lindenstrauss and Pelczyński [26], is necessary before we consider our example.

Definition IV.10. Let  $\sigma: \ell_1 \rightarrow \ell_\infty$  be the "sum operator," namely the operator mapping the sequence  $(a_i) \in \ell_1$  into the sequence of its partial sums  $(\sum_{i=1}^n a_i)_n \in \ell_\infty$ .

Theorem IV.11. Let  $E$  and  $F$  be Banach spaces and let  $T \in L(E, F)$ . The operator  $T$  is not weakly compact if and only if there exists operators  $J: \ell_1 \rightarrow X$  and  $U: Y \rightarrow \ell_\infty$  such that  $UTS = \sigma$ .

Example IV.12. Let  $D_2: \ell_\infty \rightarrow \ell_2$  be a diagonal operator corresponding to an element of  $\ell_2$ . Then the composition  $D_2\sigma: \ell_1 \rightarrow \ell_2$  is totally factorable.

Proof: First assume that  $E$  is a reflexive Banach space. Observe that the adjoint  $D_2: \ell_2 \rightarrow (\ell_\infty)'$  is compact and hence by II.7 is locally factorable. But since  $\sigma'$  maps in  $\ell_\infty$ , we can conclude that  $\sigma'D_2$  is totally factorable. Now applying IV.9  $D_2\sigma = D_2\sigma''i$  factors through every conjugate space.

Non-reflexive case: If  $E$  is non-reflexive, then the identity operator on  $E$  is not weakly compact. Hence, by IV.4  $\sigma$  factors through  $E$ . Therefore  $D_2\sigma$  factors through  $E$ .

Johnson defines the Banach space  $C_Z(E, F)$  of operators which factor compactly through  $Z$ , where  $Z = \sum_p Z$ . It can easily be shown that the nuclear operators are contained in every Banach ideal. Hence by II.25, we see that the "Little Grothendieck" conjecture is solved for spaces satisfying  $Z = \sum_p Z$ . This suggests the following question: Is the collection of all totally factorable operators a Banach ideal. If the answer to this question is yes, then we solve the conjecture. Unfortunately, even the problem of showing that the sum of two totally factorable operators is again totally factorable appears to be extremely difficult. If it were true that every Banach space  $E$  could be written as:  $E = X + Y$  with  $\dim X = \infty$  and  $\dim Y = \infty$ , then the sum problem follows immediately. The decomposition of a Banach space in this manner is a long standing problem. A result of Lindenstrauss in this direction is the following theorem.

Theorem IV.13. ([1, Page 243]) Let  $X$  be a Banach space which is generated by a weakly compact set and let  $Y$  be a separable subspace of  $X$ . Then there is a separable subspace  $Z$  of  $X$  which contains  $Y$  and a projection of norm one from  $X$  into  $Z$ .

In particular, this result shows that the sum of two totally factorable operators will factor through every non-separable reflexive space.

Grothendieck proved in [14] the existence of a nuclear operator  $T$  whose astriction is not nuclear. Our next example shows that there exists totally factorable operators whose astriction is not totally factorable.

Example IV.14. Let  $J: \ell_2 \rightarrow \ell_\infty(\Gamma)$  be an isometry, then  $iJD_\infty$  is totally factorable but not totally factorable.

$D_\infty: \ell_2 \rightarrow \ell_2$  and  $i: J(\ell_2) \rightarrow \ell_\infty(\Gamma)$ . It is clear that  $iJD_\infty$  is totally factorable since  $D_\infty$  is locally factorable and it is followed by a map into  $\ell_\infty(\Gamma)$  which has the extension property. Now the astriction of  $iJD_\infty$  is just  $JD_\infty$ . If  $JD_\infty$  is totally factorable then  $D_\infty = J^{-1}JD_\infty$  is totally factorable. But we have shown in chapter II that not every compact map on  $\ell_2$  is totally factorable.

Our final result shows the existence "in most cases" of non-totally factorable operators.

Theorem IV.15. Suppose  $E$  is a  $S_p$ -space and  $F$  is a  $S_q$ -space for  $1 < p, q < +\infty$ . Then, there exists  $T \in L(E, F)$  such that  $T$  is not toally factorable.

## REFERENCES

1. R. D. Anderson, Symposium on Infinite Dimensional Topology, Annals of Mathematics Studies, Study 69, pp. 235-273.
2. D. Amir and J. Lindenstrauss, The Structure of Weakly Compact Sets in Banach Spaces, Annals of Math. 88 (1968), pp. 35-46.
3. J. Cohen, Absolutely  $p$ -summing,  $p$ -nuclear Operators and their Conjugates, Math Ann. 201, pp. 177-200 (1973).
4. S. Banach, Théorie des opérations linéaires, Warszawa (1932).
5. M. M. Day, Normed Linear Spaces, Berlin (1962).
6. E. Dubinsky, A. Pelczyński, and H. Rosenthal, On Banach Spaces  $X$  for which  $\pi_2(L_\infty, X) = L(L_\infty, X)$ , Studia Math 54, pp. 617-648 (1972).
7. N. Dunford and J. Schwartz, Linear Operator I. Interscience, New York (1958).
8. A. Dvoretzky. Some Results on Convex Bodies in Banach Spaces, Proc. Symp. on Linear Spaces, Jerusalem (1961), pp. 123-160.
9. T. Figiel, Factorization of Compact Operators and Applications to the Approximation Problem, Studia Math 45 (1973), pp. 191-210.

10. T. Figiel, W. B. Johnson, L. Tzafriri, On Banach Lattices and Spaces Having Local Unconditional Structures, with Applications to Lorentz Function Spaces, Journal of Approximation Theory. Vol. 13, No. 4, pp. 395-412.
11. D. J. H. Garling, Lattice Bounding, Radonifying and Absolutely Summing Mappings, Proc. Cambridge Philos Soc. 77 (1974), 327-333.
12. Y. Gordon and D. R. Lewis, Absolutely Summing Operators and Local Unconditional Structures, Acta Math 133 (1974) 223-238.
13. Y. Gordon, D. R. Lewis, and J. R. Retherford, Banach Ideals of Operators with Applications, Journal of Functional Analysis 14 (1973), 85-129.
14. A. Grothendieck, Produits Tensoriels Topologiques et espaces nucléaires, Memoirs AMS 16 (1955).
15. A. Grothendieck, Résumé de la théorie métrique des produits tensoriels topologiques, Bol. Soc. Mathem., Sao Paulo 8 (1956), 1-79.
16. B. Grünbaum, Projection Constants, Tran. Amer. Math. Soc. 95 (1960), pp. 451-465.
17. V. I. Gurarii, Inclinations of Subspaces and Conditional Bases in Banach, Dokl. Akad. Nauk SSSR 145 (1962), 504-506 = Soviet Math. Dokl. 3 (1962), 1028-1030.
18. W. B. Johnson, On Finite Dimensional Subspaces of Banach Spaces with Local Unconditional Structure, Studia Math. 51 (1974), pp. 223-238.



19. W. B. Johnson and H. P. Rosenthal, On  $w^*$ -basic Sequences and their Applications to the Study of Banach Spaces, *Studia Math* 43 (1972) pp. 77-92.
20. W. B. Johnson, H. P. Rosenthal and M. Zippin, On Bases, Finite Dimensional Decompositions and Weaker Structures in Banach Spaces, *Israel Journal of Math.* Vol. 9, No. 4 (1971), pp. 488-506.
21. W. B. Johnson and L. Tzafriri, On the Local Structure of Subspaces of Banach Lattices, *Israel Journal of Mathematics*, Vol. 20, Nos. 3-4 (1975), pp. 292-299.
22. B. Josefson, Weak Sequential Convergence in the Dual of a Banach Space Does Not Imply Norm Convergence, *Bulletin of the AMS*, Vol. 81, No. 1 (1975), pp. 166-168.
23. S. Kwapień, On a Theorem of L. Schwartz and its Applications to Absolutely Summing Operators, *Studia Math* 38 (1970) 193-201.
24. S. Kwapień, On Operators Factorizable Through  $L_p$ -Spaces, *Bull. Soc. Math France Mém.* 31/32 (1972) 215-225.
25. D. R. Lewis, An Isomorphic Characterization of the Schmidt Class, to appear
26. J. Lindenstrauss and A. Pelczyński, Absolutely Summing Operators in  $L_p$ -spaces and their Applications, *Studia Math.* 29 (1968), 275-326.
27. J. Lindenstrauss and H. P. Rosenthal, The  $L_p$ -spaces, *Israel J. Math.* 7 (1969), 325-349.

28. J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces, Lecture Notes in Mathematics 338.
29. J. S. Morrell and J. R. Retherford,  $p$ -trivial Banach spaces, *Studia Math.* 43 (1972), 1-25.
30. A. Pelczyński, A Characterization of Hilbert-Schmidt Operators, *Studia Math.* 28 (1967), pp. 355-360.
31. A. Pelczyński, Projections on Certain Banach Spaces, *Studia Math.* 19 (1960) 209-228.
32. A. Persson and A. Pietsch,  $p$ -nucleare und  $p$ -integrale Abbildungen in Banach Räumen, *Studia Math.* 33 (1969), 19-62.
33. A. Pietsch, Nucleare lokalkonvexe Räume, Akademie-Verlag, Berlin (1965).
34. A. Pietsch, Absolut  $p$ -summierende Abbildungen in normierten Räume, *Studia Math.* 28 (1967), pp. 338-353.
35. J. R. Retherford, Applications of Banach Ideals of Operators, *Bull. of AMS*, Vol. 81, No. 6 (Nov. 1975), pp. 978-1012.
36. J. R. Retherford and C. P. Stegall, Fully Nuclear and Completely Nuclear Operators with Applications to  $L_1$ - and  $L_\infty$ -spaces, pp. 457-492.
37. A. Tong, Diagonal Nuclear Operators on  $\ell_p$ -spaces, *Trans. AMS* 143 (1969), 235-247.

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